Solvability for some boundary value problems with ϕ-Laplacian operators

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Abstract: We study the existence of solution for the one-dimensional ϕ-laplacian equation $(\phi(u'))' = \lambda f(t, u, u')$ with Dirichlet or mixed boundary conditions. Under general conditions, an explicit estimate λ_0 is given such that the problem possesses a solution for any $|\lambda| < \lambda_0$.

Keywords: ϕ-Laplacian, Schauder fixed point theorem, Dirichlet boundary value problem, mixed boundary value problem.

1 Introduction

In this paper we shall consider the differential equation

$$(\phi(u'))' = \lambda f(t, u, u') \quad \text{for a.a. } t \in I := [0, 1], \tag{1.1}$$

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with Dirichlet

$$u(0) = 0 = u(1) \tag{1.2}$$

or mixed boundary conditions

$$u(0) = 0 = u'(1). \tag{1.3}$$

We suppose that $\phi : (-a, a) \to (-b, b)$ is an increasing homeomorphism with $\phi(0) = 0$ and $0 < a, b \leq \infty$, $\lambda \in \mathbb{R}$ is a parameter and $f : I \times \mathbb{R} \times (-a, a) \to \mathbb{R}$ is a L^1-Carathéodory function, that is,

(i) for all $(u, v) \in \mathbb{R} \times (-a, a)$, $f(\cdot, u, v)$ is measurable;

(ii) for a.a. $t \in I$, $f(t, \cdot, \cdot)$ is continuous;

(iii) for each compact set $K \subset \mathbb{R} \times (-a, a)$ there exists $h_K(t) \in L^1(0, 1)$ such that

$$|f(t, u, v)| \leq h_K(t) \quad \text{for a.a. } t \in I \text{ and all } (u, v) \in K.$$

A *solution* of (1.1)-(1.2) or (1.1)-(1.3) is a function $u \in C^1(I)$ such that $\phi \circ u' \in W^{1,1}(0, 1)$ and u fulfills (1.1) almost everywhere and the corresponding boundary condition.

The study of the ϕ-laplacian equation is a classical topic that has attracted the attention of many researchers because of its interest in applications. Usually, a ϕ-laplacian operator is said *singular* when the domain of ϕ is finite (that is, $a < +\infty$), on the contrary the operator is said *regular*. On the other hand we say that ϕ is *bounded* if its range is finite (that is, $b < +\infty$) and *unbounded* in other case. There are three paradigmatic models in this context:

- $a = b = +\infty$ (Regular unbounded): the p-laplacian operator

$$\phi_1(x) = |x|^{p-2}x, \quad \text{with } p > 1.$$

- $a < +\infty$, $b = +\infty$ (Singular unbounded): the relativistic operator

$$\phi_2(x) = \frac{x}{\sqrt{1 - x^2}}.$$

- $a = +\infty$, $b < +\infty$ (Regular bounded): the one-dimensional mean curvature operator
$$\phi_3(x) = \frac{x}{\sqrt{1+x^2}}.$$

Among them, the p-laplacian operator has deserved a lot of attention and the number of related references is huge (see for instance ([5, 6, 7, 8, 9, 10] and references therein). For the relativistic operator, it has been proved in the recent paper [2] that the Dirichlet problem is always solvable. This is a striking result closely related with the "a priori" bound of the derivatives of the solutions. For the curvature operator, this is no longer true, but other results about existence and multiplicity of solutions can be obtained by variational [4] or topological approaches (see the thesis [3] for a more complete bibliography).

The purpose of this note is to contribute to the literature by proving the existence of solution for small λ, giving an explicit estimate. This complements in part the results in [4]. Moreover we extend some previous results of Bereanu and Mawhin [1, 2]. The proof is elementary and relies on Schauder's fixed point theorem after a suitable reduction of the problem to a first order integro-differential equation.

For convenience, for each $0 < r < b$ let M_r be defined as

$$M_r := \|h_{K_r}\|_1, \tag{1.4}$$

where $K_r = [\phi^{-1}(-r), \phi^{-1}(r)] \times [\phi^{-1}(-r), \phi^{-1}(r)]$.

2 The Dirichlet boundary value problem

Let us consider the boundary value problem

$$(\phi(u'))' = \lambda f(t, u, u') \quad \text{for a.e. } t \in I, \quad u(0) = 0 = u(1), \tag{2.5}$$

under the conditions given in the introduction.

Let us define the space

$$H = \left\{ y \in C(I) \, : \, \|y\|_\infty < \frac{b}{2} \right\}.$$

Of course, $\frac{b}{2}$ must be understood as $+\infty$ when $b = +\infty$. The following result is a slight modification of [2, Lemma 1], but we include the proof for the shake of completeness.

LEMMA 2.1 *For any $y \in H$ there exists a unique constant $\alpha := Q_\phi[y]$ such that*

$$\int_0^1 \phi^{-1}(y(s) - \alpha)ds = 0. \tag{2.6}$$

Besides, $|Q_\phi[y]| \leq \|y\|_\infty$ and the function $Q_\phi : H \to (-\frac{b}{2}, \frac{b}{2})$ is continuous.

Proof. By the properties of ϕ, it is clear that

$$\int_0^T \phi^{-1}(y(s) - \|y\|_\infty)ds \leq 0 \leq \int_0^T \phi^{-1}(y(s) + \|y\|_\infty)ds.$$

By Bolzano's theorem, there exists α verifying (2.6) with $|\alpha| \leq \|y\|_\infty$. Moreover, this constant is unique by the increasing character of ϕ^{-1}. To check the continuity assume that $\{y_n\} \subset H$ is a sequence converging to some $y \in H$. Then $Q_\phi[y_n] \to c$ (taking a subsequence if it is necessary) and by the dominated convergence theorem we have

$$\int_0^T \phi^{-1}(y(s) - c)ds = 0.$$

Therefore $c = Q_\phi[y]$ and the proof is complete. □

By means of a suitable change of variables we relate the problem (2.5) with the non-local first order equation

$$y'(t) = \lambda f\left(t, \int_0^t \phi^{-1}(y(s) - Q_\phi[y])ds, \phi^{-1}(y(t) - Q_\phi[y])\right) \text{ a.a. } t \in I. \tag{2.7}$$

LEMMA 2.2 *If y is a solution of problem (2.7) with $\|y\|_\infty < \frac{b}{2}$ then*

$$u(t) = \int_0^t \phi^{-1}(y(s) - Q_\phi[y])ds,$$

is a solution of problem (2.5).

The proof of the lemma is direct and thus we omit it. Now, we are in a position to prove the main result of this section: the solvability of problem (2.5) for small λ.

Theorem 2.1 *For each $0 < r < b/2$, let M_r be defined by equation (1.4). If*

$$|\lambda| < \lambda_0 := \sup_{0<r<b/2} \frac{r}{M_{2r}},$$

then problem (2.5) has a solution.

Proof. Let $0 < r_1 < b/2$ be such that $|\lambda| \le \frac{r_1}{M_{2r_1}}$ and consider the closed ball

$$B_{r_1} = \{y \in C(I) : \|y\|_\infty \le r_1\}.$$

For each $y \in B_{r_1}$ define the operator

$$Ty(t) := \lambda \int_0^t f\left(s, \int_0^s \phi^{-1}(y(\tau) + Q_\phi[y])d\tau, \phi^{-1}(y(s) + Q_\phi[y])\right) ds.$$

It is easy to show that T is completely continuous. Moreover by our assumptions and the choice of r_1 we have

$$\|Ty\|_\infty \le |\lambda| M_{2r_1} \le r_1,$$

which implies that $T(B_{r_1}) \subset B_{r_1}$. Thus Schauder's fixed point theorem yields a fixed point for T which is a solution of equation (2.7) and therefore by Lemma 2.2 it is also a solution for problem (2.5). $\square$

Remark 2.1 *Of course, an analogous result holds for a BVP defined on an arbitrary interval $[t_1, t_2]$, we have chosen the interval $[0, 1]$ just for simplicity.*

Remark 2.2 *In [4] the authors obtain the existence of a positive solution to problem*

$$-\phi(u')' = \lambda f(t, u), \quad u(0)=0=u(1),$$

for small and/or large $\lambda > 0$, where $\phi(u) = \frac{u}{\sqrt{1+u^2}}$ is the mean curvature operator. The main advantage of our approach is the simplicity on the assumptions and the fact that the constant λ_0 is established explicitly. Regrettably our method doesn't avoid in general the existence of the trivial solution.

Now we are going to apply Theorem 2.1 to study the solvability of the Dirichlet problem

$$(\phi(u'))' = f(t, u, u') \quad \text{for a.e. } t \in I, \quad u(0) = 0 = u(1), \tag{2.8}$$

extending some previous results in [1, 2]. We point out that problem (2.8) presents interesting different features depending on the bounded or unbounded behavior of ϕ.

2.1 Unbounded ϕ-laplacian ($b = +\infty$)

A consequence of Theorem 2.1 is that whenever ϕ is unbounded and f is L^1-bounded then (2.8) is always solvable.

COROLLARY 2.1 *Assume that ϕ is unbounded (that is, $b = +\infty$) and there exists $h \in L^1(0,1)$ such that*

$$|f(t,u,v)| \leq h(t) \quad \textit{for a.a. } t \in I \textit{ and all } u, v \in (-a,a). \tag{2.9}$$

Then the Dirichlet problem (2.8) has at least one solution.

Proof. By condition (2.9) it is clear that

$$M_r = \|h\|_1 \quad \text{for each } r > 0.$$

Therefore

$$\lambda_0 = \sup_{0<r<+\infty} \frac{r}{M_{2r}} = +\infty,$$

and then Theorem 2.1 ensures us that problem (2.5) has a solution for each $\lambda \in \mathbb{R}$, and in particular for $\lambda = 1$. □

REMARK 2.3 *Corollary 2.1 applies in particular if ϕ is also singular ($a < +\infty$) and f continuous on $I \times \mathbb{R}^2$. In this way Corollary 2.1 improves [2, Corollary 1].*

2.2 Bounded ϕ-laplacian ($b < +\infty$)

In the case of bounded ϕ-laplacian the "universal" solvability of (2.8) is not longer true even for a constant nonlinearity $f(t,u,v) \equiv M$ as we show in the following result.

Proposition 2.1 *Assume that ϕ is bounded (that is, $b < +\infty$), let $M \in \mathbb{R}$ and consider the Dirichlet problem*

$$(\phi(u'))' = M \quad \textit{for a.a. } t \in I, \quad u(0) = 0 = u(1). \tag{2.10}$$

Then the following claims hold:

(i) If $|M| \geq 2b$ then the problem (2.10) has no solution.

(ii) If $|M| < 2b$ and moreover Φ is odd then the problem (2.10) has a solution.

Proof. *(i)* If u is a solution of (2.10) then there exists some $\tau \in (0,1)$ such that $\phi(u'(\tau)) = 0$ and therefore we have

$$u(t) = \int_0^t \phi^{-1}(M(s-\tau))ds \quad \text{for all } t \in I. \tag{2.11}$$

Suppose that $|M| \geq 2b$, then

$$\max_{s \in [0,1]} |M(s-\tau)| \geq b.$$

In this case u given by (2.11) would not be well defined since the domain of ϕ^{-1} is the interval $(-b, b)$ and thus a solution of (2.10) can not exist.

(ii) If $|M| < 2b$ the function u given by equation (2.11) is well defined for $\tau = 1/2$, satisfies $(\phi(u'))' = M$ on $[0,1]$ and $u(0) = 0$. Moreover, since ϕ is odd we also obtain that $u(1) = 0$ and thus u is a solution of (2.10). □

Both claims of Proposition 2.1 apply to the one-dimensional mean curvature operator $\Phi(s) = \frac{s}{\sqrt{1+s^2}}$ since it is a bounded and odd homeomorphism.

Corollary 2.2 *The Dirichlet boundary value problem*

$$\left(\frac{u'}{\sqrt{1+u'(x)^2}}\right)' = M \quad \textit{for a.a. } t \in I, \quad u(0) = 0 = u(1),$$

has a solution if and only if $|M| < 2$.

As consequence of Theorem 2.1 we obtain the following sufficient condition for the solvability of the Dirichlet problem which extends a previous result in [1].

Corollary 2.3 *Assume that ϕ is bounded (that is, $b < +\infty$) and there exists $h \in L^1(0,1)$ such that*

$$|f(t,u,v)| \le h(t) \quad \textit{for a.a. } t \in I \textit{ and all } u, v \in (-a, a),$$

with

$$\|h\|_1 < \frac{b}{2}.$$

Then the Dirichlet problem (2.8) has at least one solution.

Proof. Now, for each $0 < r < b$ we have that $M_r = \|h\|_1 < \frac{b}{2}$. Therefore

$$\lambda_0 = \sup_{0<r<\frac{b}{2}} \frac{r}{M_{2r}} > 1,$$

and thus Theorem 2.1 implies the existence of a solution for problem (2.5) with $\lambda = 1$. $\square$

3 The mixed boundary value problem

If compared with the Dirichlet problem, the mixed boundary value problem

$$(\phi(u'))' = \lambda f(t,u,u') \quad \text{for a.a. } t \in I, \quad u(0) = 0 = u'(1), \tag{3.12}$$

is less studied in the related literature. In this case, by means of the change of variables $y = \phi(u')$ we have that a solution $u : I \to \mathbb{R}$ of (3.12) is equivalent to a solution $y : I \to (-b, b)$ of the following non-local first order terminal value problem

$$y'(t) = \lambda f\left(t, \int_0^t \phi^{-1}(y(s))ds, \phi^{-1}(y(t))\right) \quad \text{for a.a. } t \in I, \quad y(1) = 0. \tag{3.13}$$

By using the same idea as in Theorem 2.1, we can prove the following result.

Theorem 3.1 *If*

$$|\lambda| < \tilde{\lambda}_0 := \sup_{0<r<b} \frac{r}{M_r},$$

then problem (3.12) has a solution.

Proof. Let $0 < r_1 < b$ be such that $|\lambda| \leq \frac{r_1}{M_{r_1}}$, define

$$B_{r_1} = \{y \in C(I) : \|y\|_\infty \leq r_1\}$$

and apply Schauder's fixed point theorem to the completely continuous operator $T : B_{r_1} \to B_{r_1}$ defined as

$$Ty(t) := \lambda \int_t^1 f\left(s, \int_0^s \phi^{-1}(y(r))dr, \phi^{-1}(y(s))\right) ds.$$

□

Remark 3.1 *The preceding theorem is sharp in the following sense: when $\phi(x) = \frac{x}{\sqrt{1+x^2}}$ and $f(t,u,v) \equiv M$ then it is easy to show that problem (3.12) has a solution if and only if $|\lambda| < \tilde{\lambda}_0 = \sup_{0<r<1} \dfrac{r}{M_r} = \dfrac{1}{M}$.*

Again, the extremes where the BVP is defined can be chosen arbitrarily. On the other hand, let us observe that $\tilde{\lambda}_0 \geq \lambda_0$.

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