

AMANN'S THREE SOLUTIONS THEOREMS FOR SEMILINEAR OPERATOR EQUATION AND APPLICATIONS

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ABSTRACT. The celebrated Amann three solutions theorem is extended to semilinear operator equations in ordered normed spaces. The main result is used to provide sufficient conditions for the existence of multiple solutions of boundary value problems for resonant first and second order differential equations and for a class of abstract equations involving linear CMO-maps.

1. INTRODUCTION

Fixed point theory has become a crucial field of research in nonlinear functional analysis, offering valuable insights into finding solutions for nonlinear equations. Existence theorems of fixed points provide a powerful approach to establish these solutions. As a result, researchers have been actively exploring different methods, such as topological degree theory and fixed point index, to broaden the applicability of the fixed point theory approach. Comprehensive books and references, such as [7, 8, 12, 13, 22], provide valuable resources in this area.

With respect to the multiplicity of fixed points, two classic results are particularly important: Amann's three solutions theorem [1, 2] and the Leggett-Williams fixed point theorem [14]. Li and Han [16] made a remarkable attempt to unify these two theorems by combining cone theory and fixed point index theory, utilizing functionals instead of norms. This unified framework represents a significant step towards a comprehensive understanding of fixed point theory.

O'Regan and Zima [21] successfully extended the Leggett-Williams fixed point theorem [15] and its norm type versions [24] and [25] to a more general framework of semilinear operator equations, providing powerful work for studying resonant boundary value problems. Furthermore, in [5], Chu and Wang attempted to further extend the cone expansion and compression fixed point theorem in the order form to a broader framework of semilinear operator equations

$$(1) \quad Lx = Nx,$$

where L is a linear Fredholm mapping of index zero and N is a nonlinear operator. In the past years, there have been significant advancements in finding solutions

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for the semilinear equation (1) with small parameters. These developments can be found in the papers [6, 20, 23].

This paper is motivated by the previous research with the goal of extending the Amann three solutions theorem to the wider range of the semilinear operator equation. To the best of our knowledge we provide for the first time multiplicity results in the framework of the operator equation (1). In this way we are able to obtain the existence of multiple solutions for first and second order resonant BVPs (see for instance Theorems 4.2 and 4.3) for which previous multiplicity results, such as those in [1, 14, 16], are not directly applicable.

Let us recall some definitions and notations: a cone in a real normed space X is a closed, convex subset K of X such that $\alpha x \in K$ for $x \in K$ and $\alpha \geq 0$, and $K \cap (-K) = \{0\}$. It is worth noting that a partial order in normed space X , induced by a cone $K \subset X$, is defined as

$$x \leq y \iff y - x \in K.$$

A cone K is called normal if there is a number $c > 0$ such that if $0 \leq x \leq y$ then $\|x\| \leq c\|y\|$. The cone K is said to be solid if $\text{int}(K)$ is nonempty.

For $x, y \in X$ we shall write $x \not\leq y$ if $y - x \notin K$, $x < y$ if $x \leq y$ and $x \neq y$ and $x \ll y$ if $y - x \in \text{int}(K)$.

If K and K_Y are cones in the normed spaces X and Y , respectively, we say that operator $T: X \rightarrow Y$ is

- increasing iff $x \leq y \implies Tx \leq Ty$,
- strictly increasing iff $x < y \implies Tx < Ty$.

If, in addition, K_Y is solid we say that $T: X \rightarrow Y$ is strongly increasing iff

$$x < y \implies Tx \ll Ty.$$

Note that for a linear operator $\mathcal{L}$, increasing means $\mathcal{L}(K) \subset K_Y$, strictly increasing means $\mathcal{L}(K \setminus \{0\}) \subset K_Y \setminus \{0\}$ while strongly increasing means $\mathcal{L}(K \setminus \{0\}) \subset \text{int}(K_Y)$.

Finally, for every pair $x, y \in X$ with $x \leq y$, the set

$$[x, y] = \{z \in X : x \leq z \leq y\}$$

is called an order interval.

Lemma 1.1. (Amann's Three Solutions Theorem)[2] *Let K be a normal solid cone in the real normed space X , and let $u_1, v_1, u_2, v_2 \in X$ satisfy*

$$u_1 < v_1 < u_2 < v_2.$$

Suppose that $A: [u_1, v_2] \rightarrow X$ is completely continuous, strongly increasing and the following inequalities hold:

$$u_1 \leq Au_1, \quad Av_1 < v_1, \quad u_2 < Au_2, \quad Av_2 \leq v_2.$$

Then A has at least three fixed points x_1, x_2, x_3 in $[u_1, v_2]$ such that

$$u_1 \leq x_1 \ll v_1, \quad u_2 \ll x_2 \leq v_2 \quad \text{and} \quad u_2 \not\leq x_3 \not\leq v_1.$$

The rest of the paper is organized as follows: in Section 2, we discuss some preliminary results from the theory of Fredholm operators. Section 3 presents the abstract theorems, which extend Amann's three solutions theorem to the framework of semilinear operator equations. These results provide a general approach to a wide range of problems. In Section 4, we apply the theorems derived in Section 3 to establish multiplicity of solutions to first- and second-order nonlocal boundary

value problems at resonance. Finally, Section 5 deals with existence results for an abstract equation involving a class of mappings arising from [4].

2. PRELIMINARIES

For the convenience of the reader we collect here some basic definitions and properties about projectors and Fredholm mappings. A more detailed account can be found in the monographs [10, 18].

2.1. Linear projectors. Let X be a real normed space and $P : X \rightarrow X$ a linear and bounded operator. We say that P is a *projector* iff $P \circ P = P$.

Remark 2.1. It is easy to show that a bounded linear operator $P : X \rightarrow X$ is a projector if and only if $\text{Im}(P)$ is an invariant subspace for P , that is, $P(w) = w$ for all $w \in \text{Im}(P)$.

Nonlinear projectors are usually called *retractions* and some well-known examples include the projection onto the unit sphere $r : X \rightarrow X$ defined as

$$r(x) = x \quad \text{if } \|x\| \leq 1 \quad \text{and} \quad r(x) = \frac{x}{\|x\|} \quad \text{if } \|x\| > 1,$$

or the projection over a nonempty, closed and convex subset in a Hilbert space.

However, in the linear case the projectors have a very special structure given in the following result.

Lemma 2.1. *If P is a projector then $\text{Ker}(P)$ and $\text{Im}(P)$ are closed on X and*

$$X = \text{Ker}(P) \oplus \text{Im}(P).$$

Therefore, each $v \in X$ may be uniquely written as $v = u + w$ with $u \in U := \text{Ker}(P)$ and $w \in W := \text{Im}(P)$ and

$$(2) \quad P : U \oplus W \rightarrow X \quad \text{is given by} \quad P(u + w) = w.$$

Example 2.1. If X is a real normed space and $\lambda \in X^*$ is a non-zero element in the dual of X , that is, $\lambda : X \rightarrow \mathbb{R}$ is a bounded linear functional and there exists $\gamma \in X$ such that $\lambda[\gamma] \neq 0$ then

$$Px = \frac{\lambda[x]}{\lambda[\gamma]} \gamma$$

is a projector in X for which

$$\text{Ker}(P) = \{x \in X : \lambda[x] = 0\} \quad \text{and} \quad \text{Im}(P) = \{c\gamma : c \in \mathbb{R}\}.$$

Hence, from Lemma 2.1, P induces the decomposition

$$X = \{x \in X : \lambda[x] = 0\} \oplus \{c\gamma : c \in \mathbb{R}\}.$$

We will use this kind of projectors in the applications presented in Section 4.

Now, given a closed subspace $W \subset X$ two natural questions arise:

- does there exist a projector $P : X \rightarrow X$ such that $\text{Im}(P) = W$?
- does there exist a projector $Q : X \rightarrow X$ such that $\text{Ker}(Q) = W$?

From Lemma 2.1 a necessary condition for an affirmative answer to either question is that W admits a *topological complement*, that is, a closed subspace $U \subset X$ such that $X = U \oplus W$.

On the other hand, we have the following sufficient conditions:

- If X is a *Banach space* and W admits a topological complement U then there exists a projector $P : X \rightarrow X$ such that $\text{Im}(P) = W$ and $\text{Ker}(P) = U$. Indeed, just define P as in (2): clearly, P is linear and the continuity is a consequence of the open mapping theorem valid in complete normed spaces, [3, Section 2.4].
- If W is a *finite dimensional* subspace of X then there exists a projector $P : X \rightarrow X$ such that $\text{Im}(P) = W$ as a consequence of the Hahn–Banach theorem, [3, Corollary 1.2].
- If $U \subset X$ is a closed subspace and has a *finite codimension* then there exists a projector $Q : X \rightarrow X$ such that $\text{Ker}(Q) = U$. One just have to define Q as in [22, p. 369 (P)].

Remark 2.2. Note that the existence of a topological complement for a closed subspace $W \subset X$ is, in general, a delicate matter but can be guaranteed in the following simple cases, [3, Section 2.4]:

- W is a *finite dimensional* subspace.
- W has a *finite codimension*.
- X is a Hilbert space.

Finally, be aware that the existence of topological complements is also closely related to the right and left invertibility of bounded linear operators, [3, Theorems 2.12 and 2.13].

2.2. Fredholm operators. Let X and Y be real normed spaces and $L : \text{Dom}(L) \subset X \rightarrow Y$ a linear mapping such that:

- (H0) $\text{Ker}(L)$ has a finite dimension
- (H1) $\text{Im}(L)$ is closed and has finite codimension.

A mapping L satisfying (H0) and (H1) is called a *Fredholm mapping* and its *index* is the integer

$$\text{Ind}(L) = \dim(\text{Ker}(L)) - \text{codim}(\text{Im}(L)).$$

By the previous discussion, for such mappings there exist continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that the sequence

$$X \xrightarrow{P} X \xrightarrow{L} Y \xrightarrow{Q} Y$$

is exact, that is, $\text{Im}(P) = \text{Ker}(L)$ and $\text{Ker}(Q) = \text{Im}(L)$. Then,

$$X = \text{Ker}(L) \oplus \text{Ker}(P) \quad \text{and} \quad Y = \text{Im}(L) \oplus \text{Im}(Q),$$

and the restriction

$$L_P : \text{Dom}(L) \cap \text{Ker}(P) \rightarrow \text{Im}(L)$$

is an isomorphism, so its algebraic inverse $\mathcal{K}_P : \text{Im}(L) \rightarrow \text{Dom}(L) \cap \text{Ker}(P)$ is well defined.

If moreover L is of index zero, that is, if $\dim(\text{Ker}(L)) = \text{codim}(\text{Im}(L))$, then there exists a (non unique) isomorphism $J : \text{Im}(Q) \rightarrow \text{Ker}(L)$ and it is easy to see, [8, 18], that

$$L + J^{-1}P : \text{Dom}(L) \rightarrow Y$$

is an isomorphism with inverse

$$\mathcal{K}_P(I - Q) + JQ : Y \rightarrow \text{Dom}(L).$$

The following diagram, in which the different paths commute, is useful to summarize the situation. We denote by $E := \text{Dom}(L) \cap \text{Ker}(P)$:

$$\begin{array}{ccc}
 E & \xrightleftharpoons[L_P]{\mathcal{K}_P} & \text{Im}(L) \\
 \uparrow I-P & \nearrow L & \uparrow I-Q \\
 \text{Ker}(L) \oplus E & \xrightarrow{L+J^{-1}P} & \text{Im}(L) \oplus \text{Im}(Q) \\
 \downarrow P & \searrow J^{-1}P & \downarrow Q \\
 \text{Ker}(L) & \xrightleftharpoons[J]{J^{-1}} & \text{Im}(Q)
 \end{array}$$

3. MAIN RESULTS

Maintaining the notation of the previous section we obtain the following three solutions theorem in the framework of semilinear abstract equations.

Theorem 3.1. *Let $L : \text{Dom}(L) \subset X \rightarrow Y$ be a Fredholm operator of index zero, K be a normal solid cone in the normed space X and K_Y a cone in the normed space Y .*

Moreover, suppose that:

- (i) *$N : X \rightarrow Y$ is an L -completely continuous operator, that is, $QN : X \rightarrow Y$ and $\mathcal{K}_P(I - Q)N : X \rightarrow X$ are completely continuous.*
- (ii) *There exist $u_1, v_1, u_2, v_2 \in \text{Dom}(L)$, $u_1 < v_1 < u_2 < v_2$ such that*

$$Lu_1 \leq Nu_1, \quad Nv_1 < Lv_1, \quad Lu_2 < Nu_2, \quad Nv_2 \leq Lv_2.$$

- (iii) *$\mathcal{K}_P(I - Q) + JQ : Y \rightarrow X$ is increasing.*
- (iv) *$\mathcal{K}_P(I - Q)N + JQN + P : X \rightarrow X$ is strongly increasing in $[u_1, v_2]$.*

Then the equation $Lx = Nx$ has at least three solutions x_1, x_2, x_3 in $[u_1, v_2]$ such that

$$u_1 \leq x_1 \ll v_1, \quad u_2 \ll x_2 \leq v_2, \quad \text{and } u_2 \not\leq x_3 \not\leq v_1.$$

Proof. Clearly, the equation $Lx = Nx$ is equivalent to

$$Lx + J^{-1}Px = Nx + J^{-1}Px,$$

and since $H := L + J^{-1}P$ is an isomorphism, with $H^{-1} = \mathcal{K}_P(I - Q) + JQ$, we obtain

$$\begin{aligned}
 Lx = Nx &\iff x = H^{-1}(Nx + J^{-1}Px) \iff x = (\mathcal{K}_P(I - Q) + JQ)(Nx + J^{-1}Px) \\
 &\iff x = \mathcal{K}_P(I - Q)Nx + JQ Nx + Px \iff x = Ax,
 \end{aligned}$$

where $A : X \rightarrow \text{Dom}(L) \subset X$ is given by

$$A = \mathcal{K}_P(I - Q)N + JQN + P.$$

Then A is completely continuous by (i) (note that $\dim(\text{Im}(P)) < \infty$ so P is completely continuous and J is an isomorphism between spaces of finite dimension) and strongly increasing by (iv). Finally, from $Lu_1 \leq Nu_1$ and (iii), we have

$$(\mathcal{K}_P(I - Q) + JQ)(Nu_1 - Lu_1) \geq 0 \iff \mathcal{K}_P(I - Q)Nu_1 + JQ Nu_1 - (I - P)u_1 \geq 0$$

and so

$$u_1 \leq Au_1.$$

Similarly, we can prove that

$$Av_1 < v_1, \quad u_2 < Au_2, \quad Av_2 \leq v_2.$$

By Lemma 1.1, A has at least three fixed points x_1, x_2, x_3 in $[u_1, v_2]$ such that $u_1 \leq x_1 \ll v_1$, $u_2 \ll x_2 \leq v_2$ and $u_2 \not\leq x_3 \not\leq v_1$. Since $x = Ax$ is equivalent to $x \in \text{Dom}(L)$ and $Lx = Nx$, we conclude that Theorem 3.1 holds. $\square$

Remark 3.1. Note that, as $\mathcal{K}_P(I - Q) + JQ$ is an isomorphism, condition (iii) in Theorem 3.1 implies in fact that $\mathcal{K}_P(I - Q) + JQ$ is strictly increasing.

Corollary 3.1. *In the framework of Theorem 3.1 assume that (i) and (ii) are satisfied and moreover either one of the following conditions holds:*

- (v) $\mathcal{K}_P(I - Q) + JQ$ is strongly increasing, N is strictly increasing and P is increasing,
- (vi) $\mathcal{K}_P(I - Q) + JQ$ and N are increasing and P is strongly increasing.

Then the same conclusion of Theorem 3.1 holds.

Proof. It only remains to prove that condition (iv) in Theorem 3.1 is satisfied. Taking into account that if $u \in \text{int}(K)$ and $v \in K$ then $u + v \in \text{int}(K)$ it is enough to show that either $\mathcal{K}_P(I - Q)N + JQN$ or P is strongly increasing and the other one is increasing. Thus the result follows from either (v) or (vi). $\square$

4. APPLICATIONS TO BOUNDARY VALUE PROBLEMS

4.1. First order boundary value problems. We consider the following first order boundary value problem

$$(3) \quad \begin{cases} x'(t) = f(t, x(t)), & t \in [0, 1], \\ x(0) = \lambda[x], \end{cases}$$

where $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\lambda[x]$ is a bounded linear functional on $C[0, 1]$. The Riesz representation theorem ensures that λ is the Riemann-Stieltjes integral with respect to an integrator Λ of bounded variation, that is,

$$\lambda[x] = \int_0^1 x(s) d\Lambda(s), \quad \text{for all } x \in C[0, 1].$$

Furthermore, Λ is unique if we impose that it is right continuous in $[0, 1]$ and $\Lambda(0) = 0$, see [19].

Let $\mathbf{1}$ denote the constant function equal to 1 on $[0, 1]$. We assume that

$$\lambda[\mathbf{1}] = 1,$$

and thus the linear problem

$$\begin{cases} x'(t) = 0, & t \in [0, 1], \\ x(0) = \lambda[x], \end{cases}$$

has nontrivial constant solutions.

We consider the Banach spaces $X = C[0, 1]$ endowed with the norm

$$\|x\|_\infty := \max_{t \in [0, 1]} |x(t)|$$

and $Y = C[0, 1] \times \mathbb{R}$ with the norm $\|y\|_\infty + |a|$ for $(y, a) \in Y$.

Define the cones

$$K = \{x \in X : x(t) \geq 0 \text{ for } t \in [0, 1]\},$$

$$K_Y = \{(y, a) \in Y : y(t) \geq 0 \text{ for } t \in [0, 1], a \geq 0\},$$

and the linear operator $L : \text{Dom}(L) \subset X \rightarrow Y$

$$Lx := (x', x(0) - \lambda[x])$$

where $\text{Dom}(L) = C^1[0, 1] \subset X$. Note that K is a solid normal cone in X .

It is easy to see that L is a Fredholm operator of index zero, with

$$\text{Ker}(L) = \{c \cdot \mathbf{1} : c \in \mathbb{R}\},$$

$$\text{Im}(L) = \left\{ (y, -\lambda[\tilde{y}]) : y \in C[0, 1], \tilde{y}(t) = \int_0^t y(s) ds \right\},$$

$$\dim \text{Ker}(L) = \text{codim Im}(L) = 1.$$

We notice that, by means of Theorems 5.5.1 and 5.4.3 in [19] (see also [11]), we are able to give an alternative expression for $\lambda[\tilde{y}]$, namely

(4)

$$\begin{aligned} \lambda[\tilde{y}] &= \int_0^1 \left(\int_0^s y(t) dt \right) d\Lambda(s) = \tilde{y}(1)\Lambda(1) - \tilde{y}(0)\Lambda(0) - \int_0^1 \Lambda(s) d \left(\int_0^s y(t) dt \right) \\ &= \left(\int_0^1 y(s) ds \right) \Lambda(1) - \int_0^1 \Lambda(s) y(s) ds = \int_0^1 (\Lambda(1) - \Lambda(s)) y(s) ds. \end{aligned}$$

Next, we define the projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ as follows

$$Px = \int_0^1 x(s) ds \cdot \mathbf{1},$$

(5)

$$Q(y, a) = (\mathbf{0}, a + \lambda[\tilde{y}]),$$

where $\mathbf{1}$ and $\mathbf{0}$ denote the constant functions equal to 1 and 0 on $[0, 1]$, respectively.

Note that

$$\text{Im}(Q) = \{\mathbf{0}\} \times \mathbb{R},$$

so we can define an isomorphism $J : \text{Im}(Q) \rightarrow \text{Ker}(L)$ by $J(\mathbf{0}, a) = a$. Moreover, for $(y, a) \in \text{Im}(L)$, the inverse operator $\mathcal{K}_P : \text{Im}(L) \rightarrow \text{Dom}(L) \cap \text{Ker}(P)$ is given by

(6)

$$\mathcal{K}_P(y, -\lambda[\tilde{y}]) (t) = \int_0^1 k(t, s) y(s) ds,$$

where

(7)

$$k(t, s) = \begin{cases} s, & 0 \leq s \leq t \leq 1, \\ s - 1, & 0 \leq t < s \leq 1. \end{cases}$$

Now as consequence of Theorem 3.1 we obtain the following multiplicity result for problem (3).

Theorem 4.1. *Assume that:*

(HF₁) $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous;

(HF₂) There exist $u_1, v_1, u_2, v_2 \in C^1[0, 1]$ such that

$$u_1(t) < v_1(t) < u_2(t) < v_2(t) \text{ for all } t \in [0, 1],$$

and

$$\begin{cases} u_1'(t) \leq f(t, u_1(t)), & t \in [0, 1], \\ u_1(0) \leq \lambda[u_1], \\ v_1'(t) > f(t, v_1(t)), & t \in [0, 1], \\ v_1(0) \geq \lambda[v_1], \\ u_2'(t) < f(t, u_2(t)), & t \in [0, 1], \\ u_2(0) \leq \lambda[u_2], \\ v_2'(t) \geq f(t, v_2(t)), & t \in [0, 1], \\ v_2(0) \geq \lambda[v_2]; \end{cases}$$

(HF₃) $\Lambda(1) - \Lambda(s) \geq 1 - s$ for all $s \in [0, 1]$;

(HF₄) For all $u, v \in \left[\min_{t \in [0, 1]} u_1(t), \max_{t \in [0, 1]} v_2(t) \right]$, with $u < v$, holds that

$$(s + \Lambda(1) - \Lambda(s))(f(s, u) - f(s, v)) < v - u \text{ for all } s \in [0, 1].$$

Then problem (3) has at least three solutions $x_i \in C^1[0, 1]$, $i = 1, 2, 3$, satisfying

$$u_1(t) \leq x_1(t) < v_1(t) < u_2(t) < x_2(t) \leq v_2(t), \quad \forall t \in [0, 1],$$

and

$$v_1(t_*) < x_3(t_*) \text{ and } x_3(t^*) < u_2(t^*) \text{ for some } t_*, t^* \in [0, 1].$$

Proof. Define the operator $N : X \rightarrow Y$ by

$$Nx(t) = (f(t, x(t)), 0), \quad t \in [0, 1].$$

Then problem (3) is equivalent to equation (1). By condition (HF₂), we have

$$Lu_1 \leq Nu_1, \quad Nv_1 < Lv_1, \quad Lu_2 < Nu_2, \quad Nv_2 \leq Lv_2.$$

On the other hand, by (5) and (6), we obtain

$$JQ(y, a) = a + \lambda[\tilde{y}]$$

and

$$\left(\mathcal{K}_P(I - Q)(y, a) \right)(t) = \int_0^1 k(t, s)y(s)ds.$$

From (HF₁) it follows that N is continuous and bounded, that is, it maps bounded sets of X into bounded sets of Y . On the other hand the projector Q is completely continuous since $\dim(\text{Im}(Q)) < \infty$, so QN is completely continuous.

It remains to prove that $\mathcal{K}_P(I - Q)N : X \rightarrow X$ is completely continuous. Note that, for each $t \in [0, 1]$

$$\left(\mathcal{K}_P(I - Q)Nx \right)(t) = \int_0^1 k(t, s)f(s, x(s))ds,$$

$$\left(\mathcal{K}_P(I - Q)Nx \right)'(t) = f(t, x(t)) = Nx(t),$$

and $|k(t, s)| \leq 1$ for all $t, s \in [0, 1]$. Now it is a straightforward computation (by using the uniform continuity of f over compact sets and the Arzelà-Ascoli theorem) to show that $\mathcal{K}_P(I - Q)N$ is continuous and takes bounded sets into relatively compact ones.

Now, condition (iii) in Theorem 3.1 holds since, taking into account (4), (7) and (HF_3) , for each $(y, a) \in K_Y$

$$\begin{aligned} (\mathcal{K}_P(I - Q)(y, a) + JQ(y, a))(t) &= \int_0^1 k(t, s) y(s) ds + a + \int_0^1 (\Lambda(1) - \Lambda(s)) y(s) ds \\ &= a + \int_0^1 (k(t, s) + \Lambda(1) - \Lambda(s)) y(s) ds \geq 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} Ax(t) &:= (\mathcal{K}_P(I - Q)Nx + JQ Nx + Px)(t) \\ &= \int_0^1 [(k(t, s) + \Lambda(1) - \Lambda(s))f(s, x(s)) + x(s)] ds, \end{aligned}$$

which is strongly increasing in $[u_1, v_2]$. Indeed, from the definition of $k(\cdot, \cdot)$ we have that

$$s - 1 \leq k(t, s) \leq s \quad \text{for all } t, s \in [0, 1]$$

and hence, taking into account (HF_3) ,

$$(8) \quad 0 \leq k(t, s) + \Lambda(1) - \Lambda(s) \leq s + \Lambda(1) - \Lambda(s).$$

Let $x, y \in [u_1, v_2]$ be such that $x < y$, that is, $u_1(s) \leq x(s) \leq y(s) \leq v_2(s)$ for all $s \in [0, 1]$ and there exists $s_0 \in [0, 1]$ such that $x(s_0) < y(s_0)$. Note that for all $t \in [0, 1]$

$$Ax(t) - Ay(t) = \int_0^1 h(t, s) ds,$$

where

$$h(t, s) := [k(t, s) + \Lambda(1) - \Lambda(s)](f(s, x(s)) - f(s, y(s))) - (y(s) - x(s)) \text{ for all } t, s \in [0, 1].$$

Clearly, $h(t, s) = 0$ when $x(s) = y(s)$ and we are going to prove that

$$(9) \quad h(t, s) < 0 \quad \text{when } x(s) < y(s).$$

If $k(t, s) + \Lambda(1) - \Lambda(s) = 0$ then the result is clear. If not, we have from (8) that

$$k(t, s) + \Lambda(1) - \Lambda(s) > 0, \quad \frac{s + \Lambda(1) - \Lambda(s)}{k(t, s) + \Lambda(1) - \Lambda(s)} \geq 1 \text{ and thus}$$

$$\frac{s + \Lambda(1) - \Lambda(s)}{k(t, s) + \Lambda(1) - \Lambda(s)} h(t, s) \leq [s + \Lambda(1) - \Lambda(s)](f(s, x(s)) - f(s, y(s))) - (y(s) - x(s)).$$

Then (HF_4) implies (9) which leads to

$$Ax(t) - Ay(t) < 0 \quad \text{for all } t \in [0, 1],$$

that is, A is strongly increasing in $[u_1, v_2]$. Hence, all conditions of Theorem 3.1 are satisfied and the conclusion follows. $\square$

Now, we apply Theorem 4.1 to prove multiplicity results for the first-order periodic boundary value problem

$$(10) \quad \begin{cases} x'(t) = f(t, x(t)), & t \in (0, 1), \\ x(0) = x(1). \end{cases}$$

Notice that in this case

$$\lambda[x] = x(1) = \int_0^1 x(s) d\Lambda(s),$$

with

$$\Lambda(s) = \begin{cases} 0, & s \in [0, 1), \\ 1, & s = 1. \end{cases}$$

Clearly, $\lambda[1] = 1$ and (HF_3) is satisfied. Therefore we have the following multiplicity result for periodic solutions.

Theorem 4.2. *If the assumptions (HF_1) , (HF_2) and (HF_4) are satisfied, then the periodic problem (10) has at least three solutions $x_i \in C^1[0, 1]$, $i = 1, 2, 3$, satisfying*

$$u_1(t) \leq x_1(t) < v_1(t) < u_2(t) < x_2(t) \leq v_2(t), \quad \forall t \in [0, 1],$$

and

$$v_1(t_*) < x_3(t_*) \text{ and } x_3(t^*) < u_2(t^*) \text{ for some } t_*, t^* \in [0, 1].$$

Easier conditions to verify are given in this straightforward consequence.

Corollary 4.1. *Assume (HF_1) and moreover*

(HF'_2) *There exist $u_1, v_1, u_2, v_2 \in \mathbb{R}$, $u_1 < v_1 < u_2 < v_2$, such that for all $t \in [0, 1]$*

$$f(t, u_1) \geq 0, f(t, v_1) < 0, f(t, u_2) > 0, \text{ and } f(t, v_2) \leq 0,$$

(HF'_3) *There exists $\frac{\partial f}{\partial x}$ and*

$$\left| \frac{\partial f(t, x)}{\partial x} \right| < \frac{1}{2} \text{ for all } t \in [0, 1] \text{ and } u_1 < x < v_2.$$

Then problem (10) has at least three solutions x_i , $i = 1, 2, 3$, satisfying

$$u_1 \leq x_1(t) < v_1 < u_2 < x_2(t) \leq v_2, \quad \forall t \in [0, 1],$$

and

$$v_1 < x_3(t_*) < u_2 \text{ for some } t_* \in [0, 1].$$

Proof. It is a direct consequence of Theorem 4.2 with constant lower/upper solutions. Note that (HF_4) follows from the mean value theorem, $1 \leq s + \Lambda(1) - \Lambda(s) < 2$ and (HF'_3) . $\square$

4.2. Second order nonlocal boundary value problem at resonance. Consider the problem

$$(11) \quad \begin{cases} -x''(t) = f(t, x(t)), & t \in (0, 1), \\ a_1 x(0) - a_2 x'(0) = 0, & b_1 x(1) + b_2 x'(1) = \lambda[x], \end{cases}$$

where $a_1 \geq 0$, $a_2 > 0$, $b_i \geq 0$, $\varrho := a_1 b_1 + a_1 b_2 + a_2 b_1 > 0$, and again

$$\lambda[x] = \int_0^1 x(s) d\Lambda(s)$$

is a bounded linear functional on $C[0, 1]$ with Λ a function of bounded variation.

Denote by γ the unique solution of the semi-homogeneous boundary value problem associated to (11)

$$\begin{cases} -x''(t) = 0, & t \in (0, 1), \\ a_1 x(0) - a_2 x'(0) = 0, & b_1 x(1) + b_2 x'(1) = 1. \end{cases}$$

Clearly, $\gamma(t) = \frac{1}{\varrho}(a_1 t + a_2)$ and observe that $\gamma(t) > 0$. We also assume that

$$\lambda[\gamma] = 1,$$

which leads to the resonant case, that is, the problem

$$\begin{cases} -x''(t) = 0, & t \in (0, 1), \\ a_1x(0) - a_2x'(0) = 0, & b_1x(1) + b_2x'(1) = \lambda[x], \end{cases}$$

has nontrivial solutions. Consider now the homogeneous BVP

$$\begin{cases} -x''(t) = 0, & t \in (0, 1), \\ a_1x(0) - a_2x'(0) = 0, & b_1x(1) + b_2x'(1) = 0. \end{cases}$$

Its Green's function is given by

$$k(t, s) = \frac{1}{\varrho} \begin{cases} (b_1(1-t) + b_2)(a_1s + a_2), & 0 \leq s \leq t \leq 1, \\ (b_1(1-s) + b_2)(a_1t + a_2), & 0 \leq t < s \leq 1, \end{cases}$$

and note that $k(t, s) \geq 0$ for $t, s \in [0, 1]$ and $k(t, s) > 0$ for $t, s \in [0, 1)$.

Consider the Banach spaces $X = Y = C[0, 1]$ endowed with the norm

$$\|x\|_\infty = \max\{|x(t)| : t \in [0, 1]\},$$

and the cone

$$K = \{x \in X : x(t) \geq 0 \text{ for } t \in [0, 1]\}.$$

As in [9], define the linear operator

$$(12) \quad Lx(t) := x(t) - \gamma(t)\lambda[x], \quad \text{Dom}(L) = X.$$

Then it is easy to see that L is a Fredholm operator of index zero, with

$$\begin{aligned} \text{Ker}(L) &= \{x(t) = c\gamma(t) \text{ on } [0, 1] : c \in \mathbb{R}\}, \\ \text{Im}(L) &= \{y \in Y : \lambda[y] = 0\}, \\ \dim \text{Ker}(L) &= \text{codim Im}(L) = 1. \end{aligned}$$

Define the projectors $P : X \rightarrow X$ by

$$Px(t) = \frac{\gamma(t)}{\int_0^1 \gamma(s)ds} \int_0^1 x(s)ds = \frac{a_1t + a_2}{\frac{a_1}{2} + a_2} \int_0^1 x(s)ds,$$

and $Q : Y \rightarrow Y$ by

$$Qy(t) = \gamma(t)\lambda[y].$$

Define the isomorphism $J : \text{Im}(Q) \rightarrow \text{Ker}(L)$ by $Jy = My$, $M > 0$. Furthermore, for $y \in \text{Im}(L)$, the inverse operator $\mathcal{K}_P : \text{Im}(L) \rightarrow \text{Dom}(L) \cap \text{Ker}(P)$ is given by

$$\mathcal{K}_Py = y - Py.$$

We use the following notation:

$$\mathcal{K}_\Lambda(s) = \int_0^1 k(t, s)d\Lambda(t), \quad s \in [0, 1]$$

$$\mathcal{K}(s) = \int_0^1 k(t, s)dt, \quad s \in [0, 1]$$

and

$$G(t, s) = k(t, s) + \gamma(t) \left(M\mathcal{K}_\Lambda(s) - \frac{1}{\int_0^1 \gamma(s)ds} \mathcal{K}(s) \right), \quad t, s \in [0, 1].$$

Theorem 4.3. *We suppose the following assumptions:*

(HS₁) $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous,

(HS₂) There exist $u_1, v_1, u_2, v_2 \in \mathbb{R}$, $u_1 < v_1 < u_2 < v_2$, such that for all $t \in [0, 1]$

$$f(t, u_1\gamma(t)) \geq 0, f(t, v_1\gamma(t)) \leq 0, f(t, u_2\gamma(t)) \geq 0 \text{ and } f(t, v_2\gamma(t)) \leq 0,$$

and there exist $t_1, t_2 \in [0, 1]$ such that

$$f(t_1, v_1\gamma(t_1)) < 0 \quad \text{and} \quad f(t_2, u_2\gamma(t_2)) > 0.$$

(HS₃) There exists $M > 0$ such that the function

$$s \mapsto M \cdot \Lambda(s) - \frac{\rho}{\frac{a_1}{2} + a_2} \cdot s$$

is nondecreasing for $s \in [0, 1]$.

(HS₄) For $\frac{u_1 a_2}{\rho} \leq u < v \leq \frac{v_2(a_1 + a_2)}{\rho}$ and all $t, s \in [0, 1]$ we have that

$$G(t, s)(f(s, u) - f(s, v)) < \frac{a_1 t + a_2}{\frac{a_1}{2} + a_2} (v - u).$$

Then problem (11) has at least three solutions x_i , $i = 1, 2, 3$, satisfying

$$u_1\gamma(t) \leq x_1(t) < v_1\gamma(t) < u_2\gamma(t) < x_2(t) \leq v_2\gamma(t), \quad \forall t \in [0, 1],$$

and

$$v_1\gamma(t_*) < x_3(t_*) < u_2\gamma(t_*) \text{ for some } t_* \in [0, 1].$$

Proof. Observe that, as in [9], problem (11) can be reduced to the semilinear equation $Lx = Nx$ with L given by (12), that is,

$$Lx(t) = x(t) - \gamma(t)\lambda[x],$$

and

$$Nx(t) := \int_0^1 k(t, s)f(s, x(s))ds.$$

Since the integral operator N has a continuous kernel, $k(t, s)$, it is standard to prove from (HS₁) that N is completely continuous (see for instance [17, Proposition 3.2, Section V.3]). Moreover, both Q and $\mathcal{K}_P$ are bounded linear operators so condition (i) in Theorem 3.1 is satisfied.

By defining $Jy = My$, where M is given in (HS₃), we have from Lemma 5.1.14 (ii) in [19]

$$\begin{aligned} \mathcal{K}_P(I - Q)y(t) + JQy(t) &= y(t) + \gamma(t) \left(M \int_0^1 y(s)d\Lambda(s) - \frac{1}{\int_0^1 \gamma(s)ds} \int_0^1 y(s)ds \right) \\ &= y(t) + \gamma(t) \left(M \int_0^1 y(s)d\Lambda(s) - \frac{\rho}{\frac{a_1}{2} + a_2} \int_0^1 y(s)ds \right) \\ &= y(t) + \gamma(t) \left(\int_0^1 y(s)d \left(M \Lambda(s) - \frac{\rho}{\frac{a_1}{2} + a_2} s \right) \right). \end{aligned}$$

Now, from (HS₃) and $\gamma > 0$ it follows that

$$\mathcal{K}_P(I - Q)y + JQy \geq 0 \quad \text{for all } y \geq 0,$$

so condition (iii) in Theorem 3.1 holds. Moreover,

$$\begin{aligned}
Ax(t) &:= (\mathcal{K}_P(I - Q)Nx + JQNx + Px)(t) \\
&= Nx(t) + \gamma(t) \left(M \int_0^1 Nx(\tau) d\Lambda(\tau) - \frac{1}{\int_0^1 \gamma(s) ds} \int_0^1 Nx(\tau) d\tau \right) + Px(t) \\
&= \int_0^1 k(t, s) f(s, x(s)) ds + \gamma(t) \left[M \int_0^1 \left(\int_0^1 k(\tau, s) f(s, x(s)) ds \right) d\Lambda(\tau) \right. \\
&\quad \left. - \frac{1}{\int_0^1 \gamma(s) ds} \int_0^1 \left(\int_0^1 k(\tau, s) f(s, x(s)) ds \right) d\tau \right] + Px(t) \\
&= \int_0^1 k(t, s) f(s, x(s)) ds + \gamma(t) \left[M \int_0^1 \left(\int_0^1 k(\tau, s) f(s, x(s)) d\Lambda(\tau) \right) ds \right. \\
&\quad \left. - \frac{1}{\int_0^1 \gamma(s) ds} \int_0^1 \left(\int_0^1 k(\tau, s) f(s, x(s)) d\tau \right) ds \right] + Px(t) \\
&= \int_0^1 k(t, s) f(s, x(s)) ds + \gamma(t) \left[M \int_0^1 f(s, x(s)) \mathcal{K}_\Lambda(s) ds \right. \\
&\quad \left. - \frac{1}{\int_0^1 \gamma(s) ds} \int_0^1 f(s, x(s)) \mathcal{K}(s) ds \right] + Px(t) \\
&= \int_0^1 G(t, s) f(s, x(s)) ds + \frac{a_1 t + a_2}{\frac{a_1}{2} + a_2} \int_0^1 x(s) ds.
\end{aligned}$$

Therefore, condition (iv) in Theorem 3.1 follows from (HS_4) . Moreover (HS_2) , $\gamma(t) > 0$, $k(t, s) \geq 0$ for $t, s \in [0, 1]$ and $k(t, s) > 0$ for $t, s \in (0, 1)$ imply that

$$L(u_1\gamma) \leq N(u_1\gamma), \quad N(v_1\gamma) < L(v_1\gamma), \quad L(u_2\gamma) < N(u_2\gamma), \quad N(v_2\gamma) \leq L(v_2\gamma),$$

so condition (ii) in Theorem 3.1 holds for

$$u_1\gamma < v_1\gamma < u_2\gamma < v_2\gamma.$$

Thus, an application of Theorem 3.1 proves the result. $\square$

As a consequence of Theorem 4.3 we obtain the following multiplicity result.

Example 4.1. The non-local resonant problem

$$\begin{cases} -x''(t) = e^{-t} + a \sin(x(t)), & t \in (0, 1), \\ x'(0) = 0, \quad x(1) + x'(1) = \int_0^1 x(s) ds, \end{cases}$$

has for any $0 < a < 1/2$ at least two geometrically different solutions, that is, two solutions not differing in an integer multiple of 2π (notice that if $x(t)$ is a solution so is $x(t) + 2\pi$).

Indeed, this problem is an instance of (11) with the continuous nonlinearity

$$f(t, x) = e^{-t} + a \sin(x),$$

$a_1 = 0$, $a_2 = b_1 = b_2 = 1$ and $\Lambda(s) = s$.

Then, $\gamma(t) = 1$ and $\lambda[\gamma] = 1$, so it is resonant and moreover, taking $M = 1$,

$$\mathcal{K}_\Lambda(s) = \mathcal{K}(s) = \frac{3 - s^2}{2},$$

$$G(t, s) = \begin{cases} 2 - t, & 0 \leq s \leq t \leq 1, \\ 2 - s, & 0 \leq t < s \leq 1. \end{cases}$$

Clearly, (HS_3) is satisfied since $\Lambda(s) - s \equiv 0$. Moreover, for $0 < a < 1/2$ condition (HS_4) is also satisfied since $1 \leq G(t, s) \leq 2$ and $|\partial f(t, x)/\partial x| \leq a$. So, all the assumptions in Theorem (4.3) are satisfied if we take

$$u_1 := 0, v_1 := \frac{3\pi}{2}, u_2 := 2\pi, v_2 := \frac{7\pi}{2}.$$

Hence, there exist at least three solutions x_i satisfying

$$0 \leq x_1(t) < \frac{3\pi}{2} < 2\pi < x_2(t) \leq \frac{7\pi}{2}, \quad \forall t \in [0, 1],$$

and

$$\frac{3\pi}{2} < x_3(t_*) < 2\pi, \quad \text{for some } t_* \in [0, 1].$$

Clearly, x_1 and x_3 are two geometrically different solutions.

5. APPLICATIONS TO A CLASS OF ABSTRACT EQUATIONS

Following [4], let Ω be a nonempty compact metric space and let μ be a positive and bounded measure over Ω . We will consider in this section the Banach spaces of real functions $X = C(\Omega)$ and $Y = L^1(\Omega, \mu)$, both with their standard norms $\|u\|_\infty$ and $\|f\|_1$, respectively. We also consider in X and Y the cones of the nonnegative functions everywhere and a.e., respectively, that is

$$K := \{f \in C(\Omega) : f(t) \geq 0 \text{ for all } t \in \Omega\}$$

and

$$K_Y := \{f \in L^1(\Omega, \mu) : f(t) \geq 0 \text{ for a.e. } t \in \Omega\}.$$

Note that K is a normal solid cone in $C(\Omega)$ while K_Y is also normal but it has empty interior in $L^1(\Omega, \mu)$.

Also, we define

$$\bar{f} := \frac{1}{\mu(\Omega)} \int_\Omega f d\mu, \tilde{f} := f - \bar{f}, \tilde{Y} = \{f \in Y : \tilde{f} = 0\} \text{ and } \tilde{X} = \{u \in X : \tilde{u} = 0\}.$$

We will say that a linear operator $L : \text{Dom}(L) \subset X \rightarrow Y$ is a CMO-map if it satisfies:

(CMO₁) $\text{Ker}(L)$ is the set of constant functions,

(CMO₂) $\text{Im}(L) = \tilde{Y}$,

(CMO₃) for each $\tilde{f} \in \text{Im}(L)$ the unique solution $\tilde{u} \in \tilde{X}$ of the equation $Lu = \tilde{f}$ satisfies

$$\|\tilde{u}\|_\infty \leq c \|\tilde{f}\|_1,$$

where $c > 0$ only depends upon L .

Note that (CMO₁) and (CMO₂) imply that L is a Fredholm operator of index zero and that we can choose as bounded projectors

$$Pu = \frac{1}{\mu(\Omega)} \int_\Omega u d\mu \cdot \mathbf{1} \quad \text{for } u \in X \text{ and } Qf = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \cdot \mathbf{1} \quad \text{for } f \in Y$$

where $\mathbf{1}$ is the constant function equal to 1 on Ω .

Therefore, (CMO₃) is equivalent to the continuity of $\mathcal{K}_P : \tilde{Y} \rightarrow \tilde{X}$, that is,

$$\|\mathcal{K}_P(\tilde{f})\|_\infty \leq c \|\tilde{f}\|_1 \text{ for } \tilde{f} \in \tilde{Y}.$$

Notice that the previous inequality implies that for each $f \in Y$

$$(13) \quad |(\mathcal{K}_P(I - Q)f)(t)| \leq \|\mathcal{K}_P(I - Q)f\|_\infty \leq c\|(I - Q)f\|_1 \leq 2c\|f\|_1 \text{ for all } t \in \Omega.$$

Furthermore, provided that $f \neq 0$, we have

$$\|(I - Q)f\|_1 < 2\|f\|_1$$

and then for each $f \in Y \setminus \{0\}$ holds that

$$|(\mathcal{K}_P(I - Q)f)(t)| < 2c\|f\|_1 \text{ for all } t \in \Omega,$$

that is,

$$(14) \quad \|\mathcal{K}_P(I - Q)f\|_\infty < 2c\|f\|_1 \text{ for } f \in Y \setminus \{0\}.$$

Theorem 5.1. *Let L be a CMO-map, $N: X \rightarrow Y$ and suppose there exist*

$$u_1 < v_1 < u_2 < v_2 \in \text{Dom}(L),$$

such that

$$Lu_1 \leq Nu_1, \quad Nv_1 < Lv_1, \quad Lu_2 < Nu_2, \quad Nv_2 \leq Lv_2.$$

If moreover

(HN₁) N is completely continuous,

$$(HN_2) \quad Nu - Nv < \frac{1}{4c\mu(\Omega)}(v - u) \text{ for } u_1 \leq u < v \leq v_2,$$

then the equation $Lx = Nx$ has at least three solutions x_1, x_2, x_3 in $[u_1, v_2]$ such that

$$u_1 \leq x_1 \ll v_1, \quad u_2 \ll x_2 \leq v_2, \quad \text{and } u_2 \not\leq x_3 \not\leq v_1.$$

Proof. We have to check that the assumptions of Theorem 3.1 hold. Firstly, N is L -completely continuous because of (HN_1) and the fact that Q and $\mathcal{K}_P$ are bounded linear operators.

By (13) and (14) we have that

$$(15) \quad \mathcal{K}_P(I - Q)f \geq -2c \int_\Omega |f| d\mu \cdot \mathbf{1} \quad \text{for all } f \in Y,$$

and

$$(16) \quad \mathcal{K}_P(I - Q)f \gg -2c \int_\Omega |f| d\mu \cdot \mathbf{1} \quad \text{for all } f \in Y \setminus \{0\}.$$

Now, define the isomorphism $Ju = Mu$ with $M = 2c\mu(\Omega)$. If $f \geq 0$ then from (15) it follows

$$\mathcal{K}_P(I - Q)f + JQf \geq -2c \int_\Omega f d\mu \cdot \mathbf{1} + \frac{M}{\mu(\Omega)} \int_\Omega f d\mu \cdot \mathbf{1} = \mathbf{0},$$

so $\mathcal{K}_P(I - Q) + JQ$ is increasing.

Finally, condition (iv) in Theorem 3.1 follows from the previous fact, (HN_2) and (16). Indeed, for $u_1 \leq u < v \leq v_2$ we have

$$\begin{aligned} & [\mathcal{K}_P(I - Q) + JQ](Nv - Nu) + P(v - u) \\ & \geq [\mathcal{K}_P(I - Q) + JQ] \left(-\frac{1}{4c\mu(\Omega)}(v - u) \right) + P(v - u) \\ & \gg -2c \frac{1}{4c\mu(\Omega)} \int_\Omega (v - u) d\mu - \frac{M}{\mu(\Omega)} \frac{1}{4c\mu(\Omega)} \int_\Omega (v - u) d\mu + \frac{1}{\mu(\Omega)} \int_\Omega (v - u) d\mu \\ & = -\frac{1}{2\mu(\Omega)} \int_\Omega (v - u) d\mu - \frac{1}{2\mu(\Omega)} \int_\Omega (v - u) d\mu + \frac{1}{\mu(\Omega)} \int_\Omega (v - u) d\mu = \mathbf{0}. \end{aligned}$$

□

Remark 5.1. In [4], there were presented several examples of CMO-maps that naturally appear in the context of differential, both ordinary and partial, and difference boundary value problems.

Note also that condition (HN_2) in Theorem 5.1 is always satisfied when N is increasing.

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