LYAPUNOV STABILITY FOR REGULAR EQUATIONS AND APPLICATIONS TO THE LIEBAU PHENOMENON

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ABSTRACT. We study the existence and stability of periodic solutions of two kinds of regular equations by means of classical topological techniques like the Kolmogorov-Arnold-Moser (KAM) theory, the Moser twist theorem, the averaging method and the method of upper and lower solutions in the reversed order. As an application, we present some results on the existence and stability of T-periodic solutions of a Liebau-type equation.

1. INTRODUCTION

Let us consider the differential equation

$$\ddot{x} + f(t,x) = 0, \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, T-periodic in the first variable and smooth enough, for example $f \in C^{0,4}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R})$. It is known that when a T-periodic solution x of (1.1) is of the twist type then it is Lyapunov stable (see Section 2 for more details). This means that the so-called first twist coefficient of the Birkhoff normal form does not vanish and an explicit expression for that coefficient in terms of the third order approximation

$$\ddot{y} + a(t)y + b(t)y^2 + c(t)y^3 + o(y^3) = 0, \tag{1.2}$$

where

$$a(t) = f_x(t,x(t)), \quad b(t) = \frac{1}{2} f_{xx}(t,x(t)), \quad c(t) = \frac{1}{6} f_{xxx}(t,x(t)).$$

was firstly obtained by Ortega [15] (see also [13, 21]) .

Some applications of Ortega's works can be found in [1, 2, 4]. In a recent work [3], based on the above ideas, the existence of Lyapunov stable periodic solutions for the combined attractive-repulsive singularity has been studied by the first author, Chu and Torres. For further results on mathematical models with singularities, we refer to the recent book [19], and the references therein. However, up to now, there are few results about the existence of periodic solutions of the regular systems

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[10, 16]. The main aim of this paper is to develop some new existence and stability criteria so that the regular case can be studied following the ideas in [3] and an application to the Liebau phenomenon could be obtained. More precisely, we will study the existence and stability of T-periodic solutions of the regular equation

$$\ddot{x} = r(t)x^{\alpha} - s(t)x^{\beta}, \tag{1.3}$$

and the equation with a small parameter ε

$$\ddot{x} = r(t)x^{\alpha} - \varepsilon s(t)x^{\beta}, \tag{1.4}$$

where $0 < \alpha < \beta < 1$, and r, s are continuous and T-periodic functions.

The Liebau phenomenon or "valveless pumping effect" is referred to a preferential direction of the flow obtained in mechanical systems without valves as consequence of asymmetric periodic oscillations, [14]. A simple model which shows this effect, the so called "one pipe-one tank" configuration, was presented in [17] and a detailed description of it is also available in [5] and [19, Chapter 8]. That model leads to the searching of positive T-periodic solutions for the following singular second-order differential equation

$$\ddot{u} + a\dot{u} = \frac{1}{u}(e(t) - b\dot{u}^2) - c, \tag{1.5}$$

where

$$a = \frac{r_0}{\rho}, \quad b = 1 + \frac{\zeta}{2}, \quad c = \frac{gA_P}{A_T}, \quad e(t) = \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho},$$

and the meaning of the involved functions and parameters are the following

$r_0 \geq 0$	friction coefficient on the tube
$\rho > 0$	density of the fluid
$\zeta \geq 1$	junction coefficient
g	acceleration of gravity
$A_P > 0$	cross section of the pipe
$A_T > 0$	cross section of the tank
$V_0 > 0$	total volume of the fluid
$p(t)$	T-periodic forcing

So, from the physical point of view, it is natural to assume in equation (1.5) that

$$a \geq 0, \quad b > 1, \quad c > 0 \quad \text{and} \quad e \text{ is continuous and } T-\text{periodic}.$$

Notice that we could even assume that $b \geq 3/2$.

Recently, some results on the existence and stability of periodic positive solutions for (1.5) with friction were presented in [5, 6, 7, 19]. Furthermore, Liao obtained in [12] the existence of T-periodic solutions of a generalized Liebau-type differential equation by using the fixed point theorem in cones. However, up to now, there are few works on the conservative case $a = 0$, which is an idealization but it is interesting from both the mathematical and the physical point of view. Our results in the present paper shall fill partially this gap.

So, let us consider equation (1.5) without friction, that is,

$$\ddot{u} = \frac{1}{u}(e(t) - b\dot{u}^2) - c. \tag{1.6}$$

By means of the change of variables $u = x^\kappa$, where $\kappa = 1/(b+1)$ (see [5]), we rewrite the singular problem (1.6) as

$$\ddot{x} = \frac{e(t)}{\kappa}x^{1-2\kappa} - \frac{c}{\kappa}x^{1-\kappa}, \tag{1.7}$$

which is a regular equation in the form (1.3). Notice that it has physical sense to consider c as a small parameter, meaning that the section of the pipe is much less than the section of the tank, and then equation (1.7) fits also in the form (1.4).

The paper is organized as follows: after this Introduction, in Section 2 for the convenience of the reader we collected some general results, well known by the specialists, used in order to proof our main theorems. Section 3 is devoted to our main existence and stability criteria. The importance of such result relies in that, for the first time in this topic, we have stability theorems for the regular equations (1.3) and (1.4). In Section 4 the previous results are applied to the Liebau model (1.6) and some illustrative examples are given.

Throughout this paper the following notations will be used. For a given T-periodic continuous function h we denote

$$\bar{h} = \frac{1}{T}\int_0^T h(t)dt, \quad h_m = \min_{t\in[0,T]} h(t), \quad h_M = \max_{t\in[0,T]} h(t), \quad \tilde{h}_m = \min_{t\in[0,T]} |h(t)|,$$

and $\triangle_h = \frac{h_M}{h_m}$. This quantity $\triangle_h \geq 1$ can be regarded as a measure of the ratio of h_M to h_m and will play a key role in our main results. Furthermore, we define $\gamma = \frac{1}{\beta-\alpha}$. Since $0 < \alpha < \beta < 1$, we have $\gamma \in (1, \infty)$.

2. Preliminaries

The linearized equation of (1.2) is the Hill's equation

$$\ddot{y} + a(t)y = 0. \tag{2.1}$$

We say that (2.1) is elliptic, or linearly stable, if its Floquet multipliers λ_1, λ_2 satisfy $\lambda_1 = \overline{\lambda}_2$, $|\lambda_1| = 1$, $\lambda_1 \neq \pm 1$. In this case the rotation number ρ is defined by the relation $\lambda = \exp(\pm \mathrm{i}\rho T)$, and for convenience we write $\theta = \rho T$.

It is well known that the linear stability of (2.1) is not enough to ensure the Liapunov stability of (1.2). The T-periodic solution $x(t)$ of (1.1) is called 4-elementary if the multipliers λ of (2.1) satisfy $\lambda^q \neq 1$ for $1 \leq q \leq 4$. If $x(t)$ is 4-elementary then we say that $x(t)$ is of the twist type if the first twist coefficient μ of the Birkhoff normal form of the Poincaré map is non-zero. In that case Moser's invariant curve theorem can be applied to show that a solution of twist type is Lyapunov stable (see §32-§34 in [18]) and, moreover, around it the complex dynamics prescribed by KAM theory arises. Clearly, a major difficulty in this approach is the computation of the corresponding first twist coefficient μ. This task was firstly accomplished by Ortega in [15] where he found an explicit formula for μ, later reformulated in [21] (see also [13]) as

$$\mu = \iint_{[0,T]^2} b(t)b(\tau)R^3(t)R^3(\tau)\chi_\theta(|\varphi(t) - \varphi(\tau)|)dtd\tau - \frac{3}{8}\int_0^T c(t)R^4(t)dt, \tag{2.2}$$

where R and φ denote the polar coordinates, $\Psi(t) = R(t)\exp(\mathrm{i}\varphi(t))$ is the complex solution of (2.1) with initial conditions $\Psi(0) = 1, \Psi'(0) = \mathrm{i}$ and the kernel χ_θ is given by

$$\chi_\theta(x) = \frac{3\cos(x-\theta/2)}{16\sin(\theta/2)} + \frac{\cos 3(x-\theta/2)}{16\sin(3\theta/2)}, \qquad x \in [0,\theta].$$

2.1. **Existence lemmas.**

Definition 2.1. *A function $\sigma_1 \in C^2([0,T])$ is said to be a* **lower solution** *of (1.1) if*
(i) $\ddot{\sigma_1} + f(t,\sigma_1) \geq 0$ *for all* $t \in [0,T]$
(ii) $\sigma_1(0) = \sigma_1(T), \dot{\sigma_1}(0) \geq \dot{\sigma_1}(T)$.

Analogously, an **upper solution** σ_2 is defined by reversing the respective inequalities in the previous definition. A lower solution (resp. upper solution) is called strict if the inequality in (i) is strict. It is well known that the classical method of lower and upper solutions is a quite effective and flexible tool for studying existence of T-periodic solutions of (1.1). Actually, a couple of upper and lower solutions such that $\sigma_1(t) \leq \sigma_2(t)$ for all t typically leads to unstable solutions lying between σ_1 and σ_2 (see [9]). In order to obtain a stable solution, we now assume that σ_1 and σ_2 are ordered in the reversed way, provided the partial derivative of f with respect to x is not too large. The following result is a consequence of [8, Theorem 3.7].

Lemma 2.2. *Assume that there exist upper and lower solutions of* (1.1) *such that* $\sigma_2(t) \leq \sigma_1(t)$ *for all* t. *Under the assumption*

$$f_x(t,x) \leq \frac{\pi^2}{T^2}, \qquad \text{for any } x \in [\sigma_2(t), \sigma_1(t)],$$

equation (1.1) has a T-periodic solution x such that

$$\sigma_2(t) \leq x(t) \leq \sigma_1(t)$$

for every t.

Remark 2.3. *Under the assumptions of Lemma 2.2 all the T-periodic solutions of (1.1) between σ_2 and σ_1 are ordered and there exist the maximum and the minimum of such solutions, see* [8, Exercise 3.5].

The following lemma summarises the averaging method and provides the existence of periodic solutions on differential equations like (1.4) containing a small parameter (for more details see [11, Chapter V, Theorem 3.2]).

Lemma 2.4. *We consider the following differential system*

$$\dot{x} = \varepsilon f(t,x,\varepsilon), \tag{2.3}$$

where $f : \mathbb{R} \times D \times [0,+\infty) \to \mathbb{R}^n$ is a continuous function, T-periodic in the first variable, of class C^1 in x, ε and D is an open subset of $\mathbb{R}^n$. We define

$$f_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,x,0)dt,$$

and assume that for $x_0 \in D$ with $f_0(x_0) = 0$ we have the determinant of the Jacobian matrix $J_{f_0}(x_0) \neq 0$. Then there exist $\varepsilon_0 > 0$ and a function $x(t,\varepsilon)$, continuous for $(t,\varepsilon) \in \mathbb{R} \times [0,\varepsilon_0]$, such that $x(t,\varepsilon)$ is a T-periodic solution of (2.3) *for any $\varepsilon \in [0,\varepsilon_0]$ and $x(t,0) = x_0$. Moreover, this solution $x(t,\varepsilon)$ is unique in a neighborhood of x_0.*

2.2. **Stability lemmas.**

Lemma 2.5. [20, Theorem 3.1, Theorem 3.2] *Assume that there exists a T-periodic solution x of* (1.1) *such that:*

(i) $0 < a_m \le a_M < (\frac{\pi}{2T})^2$,
(ii) $c_m > 0$,

and either

(iii) $10\tilde{b}_m^2 a_m^{3/2} > 9c_M(a_M)^{5/2}$

or

(iii') $10\tilde{b}_M^2 a_M^{3/2} < 9c_m(a_m)^{5/2}$

is satisfied. Then the solution x is of the twist type and the Moser twist theorem [18] *implies that such a solution is stable.*

The following asymptotic behavior of R and the rotation number $\rho \equiv \rho(a)$ are very useful in order to get the sign of the first twist coefficient.

Lemma 2.6. [4, Corollary 4.1] *Assume that a in* (2.1) *is nonnegative and has a positive mean $\bar{a} > 0$. Then $R(t) =: R(t,a)$ and $\theta =: \theta(a)$ in formula* (2.2) *satisfy the asymptotic behavior*

$$R(t) = \bar{a}^{-1/4}(1 + O(\bar{a})), \quad \theta(a) = T\bar{a}^{1/2}(1 + O(\bar{a})), \quad \textit{when} \quad \bar{a} \to 0^+.$$

3. Main results

3.1. **Existence results.** Let us define

$$\Delta := \Delta_r \Delta_s = \frac{r_M s_M}{r_m s_m}, \tag{3.1}$$

$$f(t,x) = s(t)x^\beta - r(t)x^\alpha, \tag{3.2}$$

and

$$g(x) = \beta s_M x^{\beta-1} - \alpha r_m x^{\alpha-1} \quad \text{for } x > 0. \tag{3.3}$$

Notice that $\Delta \ge 1$.

Lemma 3.1. *Assume that r, s are positive T-periodic continuous functions. Then the following claims hold:*

(i) *The functions*

$$\sigma_2(t) \equiv \left(\frac{r_m}{s_M}\right)^\gamma \le \sigma_1(t) \equiv \left(\frac{r_M}{s_m}\right)^\gamma,$$

are constant upper and lower solutions of (1.3), respectively.

(ii) *For all $t \in \mathbb{R}$ and $x > 0$,*

$$f_x(t,x) \le g(x).$$

(iii) *If $\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \le 1$ then*

$$\max_{\sigma_2 \le x \le \sigma_1} g(x) = g(\sigma_2) = \frac{r_m}{\gamma}\left(\frac{s_M}{r_m}\right)^{(1-\alpha)\gamma}.$$

(iv) *If* $1 < \frac{\alpha(1-\alpha)}{\beta(1-\beta)} < \Delta$ *then*

$$\max_{\sigma_2 \le x \le \sigma_1} g(x) = g(x_0) = \frac{\alpha r_m}{\gamma(1-\beta)} \left(\frac{\beta(1-\beta)s_M}{\alpha(1-\alpha)r_m} \right)^{(1-\alpha)\gamma},$$

where

$$x_0 = \left(\frac{\alpha(1-\alpha)r_m}{\beta(1-\beta)s_M} \right)^{\gamma}.$$

(v) *If* $\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \ge \Delta$ *then*

$$\max_{\sigma_2 \le x \le \sigma_1} g(x) = g(\sigma_1) = r_m(\beta\Delta - \alpha) \left(\frac{s_m}{r_M} \right)^{(1-\alpha)\gamma}.$$

Proof. It is easy to check that (i) holds and that for all $t \in \mathbb{R}$ and $x > 0$ we have

$$\begin{aligned} f_x(t,x) &= \beta s(t) x^{\beta-1} - \alpha r(t) x^{\alpha-1} \\ &\le \beta s_M x^{\beta-1} - \alpha r_m x^{\alpha-1} = g(x), \end{aligned}$$

so (ii) holds too.

Now, $g'(x) = 0$ if and only if $x = x_0$, as defined in (iv), and g is increasing on $(0, x_0)$ and decreasing on $(x_0, +\infty)$. Therefore, either $x_0 \le \sigma_2$, $x_0 \in (\sigma_2, \sigma_1)$, or $x_0 \ge \sigma_1$ if either $\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \le 1$, $1 < \frac{\alpha(1-\alpha)}{\beta(1-\beta)} < \Delta$, or $\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \ge \Delta$ are satisfied, respectively. Thus claims (iii), (iv) and (v) also hold. □

As consequence of Lemmas 3.1 and 2.2 we have the following existence result.

Theorem 3.2. *Assume that r, s are positive T-periodic continuous functions and one of following conditions holds:*

(C1) $\frac{\alpha(1-\alpha)}{\beta(1-\beta)} < 1$ *and* $\frac{r_m}{\gamma} \left(\frac{s_M}{r_m} \right)^{(1-\alpha)\gamma} \le \left(\frac{\pi}{T} \right)^2$,

(C2) $1 \le \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \le \Delta$ *and* $\frac{\alpha r_m}{\gamma(1-\beta)} \left(\frac{\beta(1-\beta)s_M}{\alpha(1-\alpha)r_m} \right)^{(1-\alpha)\gamma} \le \left(\frac{\pi}{T} \right)^2$,

(C3) $\frac{\alpha(1-\alpha)}{\beta(1-\beta)} > \Delta$ *and* $r_m(\beta\Delta - \alpha) \left(\frac{s_m}{r_M} \right)^{(1-\alpha)\gamma} \le \left(\frac{\pi}{T} \right)^2$.

Then equation (1.3) has at least one T-periodic solution x such that

$$\left(\frac{r_m}{s_M} \right)^{\gamma} \le x(t) \le \left(\frac{r_M}{s_m} \right)^{\gamma}. \tag{3.4}$$

Corollary 3.3. *Assume that r, s are positive T-periodic continuous functions and the following inequality holds*

$$\frac{(r_m)^{\gamma(\beta-1)}}{(s_M)^{\gamma(\alpha-1)}} (\beta\Delta - \alpha) < \left(\frac{\pi}{T} \right)^2. \tag{3.5}$$

Then equation (1.3) has at least one T-periodic solution x such that (3.4) holds.

Proof. Taking into account that

$$\begin{aligned}\max_{x\in[\sigma_2,\sigma_1]} g(x) &= \max_{x\in[\sigma_2,\sigma_1]}[\beta s_M x^{\beta-1} - \alpha r_m x^{\alpha-1}] \\ &= \max_{x\in[\sigma_2,\sigma_1]}[x^{\alpha-1}(\beta s_M x^{\beta-\alpha} - \alpha r_m)] \\ &\le \left(\frac{r_m}{s_M}\right)^{\gamma(\alpha-1)}[\beta s_M \left(\frac{r_M}{s_m}\right)^{\gamma(\beta-\alpha)} - \alpha r_m] \\ &= \left(\frac{r_m}{s_M}\right)^{\gamma(\alpha-1)}[\beta s_M \left(\frac{r_M}{s_m}\right) - \alpha r_m] \\ &= \frac{(r_m)^{\gamma(\beta-1)}}{(s_M)^{\gamma(\alpha-1)}}(\beta\Delta - \alpha),\end{aligned}$$

from Lemma 3.1 and condition (3.5) it follows that Theorem 3.2 applies. □

We complete this section with the application of Lemma 2.4 to equation (1.4).

Theorem 3.4. *Assume that r and s are T-periodic continuous functions, and $\bar{r}\cdot\bar{s} > 0$. Then equation (1.4) has a T-periodic solution $x(t,\varepsilon)$ if $\varepsilon > 0$ is small enough. Moreover, the following asymptotic behavior holds*

$$\lim_{\varepsilon\to 0^+} \varepsilon^\gamma x(t,\varepsilon) = \omega^\gamma, \qquad \textit{uniformly in } t, \tag{3.6}$$

where

$$\omega = \frac{\bar{r}}{\bar{s}}.$$

Proof. To apply Lemma 2.4, first we rewrite equation (1.4) as the differential system

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= r(t)x^\alpha - \varepsilon s(t)x^\beta.\end{aligned} \tag{3.7}$$

Doing the rescaling of variables

$$\begin{aligned}x &= u\varepsilon^{-\gamma}, \\ y &= v\varepsilon^{-\frac{\gamma(\alpha+1)}{2}}, \\ \nu &= \varepsilon^{\frac{\gamma(1-\alpha)}{2}},\end{aligned}$$

system (3.7) takes the form

$$\begin{aligned}\dot{u} &= \nu v, \\ \dot{v} &= \nu\left(r(t)u^\alpha - s(t)u^\beta\right).\end{aligned} \tag{3.8}$$

The averaged system of (3.8) is

$$\begin{aligned}\dot{\xi} &= \nu\eta, \\ \dot{\eta} &= \nu\left(\bar{r}\xi^\alpha - \bar{s}\xi^\beta\right).\end{aligned} \tag{3.9}$$

After some calculations we have that the averaged system (3.9) has a unique non-trivial constant solution $(\xi_0,\eta_0) = (\omega^\gamma, 0)$, and that the determinant of the Jacobian matrix evaluated at (ξ_0,η_0) does not vanish. Then by Lemma 2.4, the equilibrium (ξ_0,η_0) is continuable for small ν, that is, there exists ν_0 such that system (3.8) has a T-periodic solution $(u(t,\nu), v(t,\nu))$ for $0 < \nu < \nu_0$, tending uniformly to (ξ_0,η_0) as $\nu \to 0^+$. Going back through the rescaling, we conclude that equation (1.4) has a T-periodic solution $x(t,\varepsilon)$ for $\varepsilon > 0$ small enough. Moreover, the asymptotic behavior (3.6) occurs. □

Remark 3.5. *The asymptotic behavior* (3.6) *will be the key in order to prove stability results in the next section. Moreover,* (3.6) *implies that the solution* $x(t,\varepsilon)$ *is positive for all* t *and* $\varepsilon > 0$ *small enough, which is essential for the application to the Liebau model.*

3.2. **Stability results.** Let x be a T-periodic solution of equation (1.3). A computation of the coefficients in (1.2) for that equation gives:

$$a(t) = \beta s(t)x(t)^{\beta-1} - \alpha r(t)x(t)^{\alpha-1},$$

$$b(t) = \frac{1}{2}\left[\beta(\beta-1)s(t)x(t)^{\beta-2} - \alpha(\alpha-1)r(t)x(t)^{\alpha-2}\right],$$

$$c(t) = \frac{1}{6}\left[\beta(\beta-1)(\beta-2)s(t)x(t)^{\beta-3} - \alpha(\alpha-1)(\alpha-2)r(t)x(t)^{\alpha-3}\right].$$

Whenever a localization for the solution x is available we can obtain some estimates for the previous coefficients. We will derive carefully those estimates in the following three lemmas.

Lemma 3.6. *Assume that* r, s *are positive* T*-periodic continuous functions and that* x *is a solution of equation (1.3) satisfying* (3.4) *(for instance if the conditions of Theorem 3.2 are fulfilled). If moreover*

(S) $\Delta < \frac{\beta(1-\beta)(2-\beta)}{\alpha(1-\alpha)(2-\alpha)}$

is satisfied, then we have:

(i) $0 < \frac{r_M}{\gamma}\left(\frac{s_m}{r_M}\right)^{\gamma(1-\alpha)} \leq a_m \leq a_M \leq \frac{r_m}{\gamma}\left(\frac{s_M}{r_m}\right)^{\gamma(1-\alpha)}$.

(ii) $c_m > 0$.

(iii) $b_M < 0$, $\widetilde{b}_m \geq \frac{1}{2}\left(\frac{r_M}{s_m}\right)^{\gamma(\alpha-2)} r_M(\beta(1-\beta)-\alpha(1-\alpha)) > 0$, *and* $\widetilde{b}_M \leq \frac{1}{2}\left(\frac{r_m}{s_M}\right)^{\gamma(\alpha-2)} r_m(\beta(1-\beta)-\alpha(1-\alpha))$.

Proof. Take into account that since $\alpha < \beta$ and $\Delta \geq 1$, assumption (S) implies

$$1 \leq \Delta < \frac{\beta(1-\beta)}{\alpha(1-\alpha)}. \tag{3.10}$$

Claim (i).- Note that $a(t) = f_x(t, x(t))$. Hence, from (3.4), (3.10) and Lemma 3.1 we obtain

$$a_M \leq \frac{r_m}{\gamma}\left(\frac{s_M}{r_m}\right)^{(1-\alpha)\gamma}. \tag{3.11}$$

On the other hand, using (3.4), (3.10) and reasoning as in the proof of Lemma 3.1 we get

$$
\begin{aligned}
a(t) &\geq \beta s_m x(t)^{\beta-1} - \alpha r_M x(t)^{\alpha-1} \\
&\geq \beta s_m \sigma_1^{\beta-1} - \alpha r_M \sigma_1^{\alpha-1} \\
&= \sigma_1^{\alpha-1}\left[\beta s_m \sigma_1^{\beta-\alpha} - \alpha r_M\right] \\
&= \left(\frac{r_M}{s_m}\right)^{\gamma(\alpha-1)}\left[\beta s_m \left(\frac{r_M}{s_m}\right) - \alpha r_M\right] \\
&= \left(\frac{r_M}{s_m}\right)^{\gamma(\alpha-1)} \frac{r_M}{\gamma} \\
&> 0.
\end{aligned}
$$

Claim (ii).- From (S) and (3.4) it follows that

$$
\begin{aligned}
c(t) &= \frac{1}{6}x(t)^{\alpha-3}\left[\beta(\beta-1)(\beta-2)s(t)x(t)^{\beta-\alpha} - \alpha(\alpha-1)(\alpha-2)r(t)\right] \\
&\geq \frac{1}{6}x(t)^{\alpha-3}[\beta(\beta-1)(\beta-2)s_m\left(\frac{r_m}{s_M}\right)^{\gamma(\beta-\alpha)} - \alpha(\alpha-1)(\alpha-2)r_M] \\
&= \frac{1}{6}x(t)^{\alpha-3}r_M[\beta(\beta-1)(\beta-2)\frac{1}{\Delta} - \alpha(\alpha-1)(\alpha-2)] \\
&> 0.
\end{aligned}
$$

Claim (iii).- Now, from (3.4) and (3.10) we deduce

$$
\begin{aligned}
b(t) &\leq \frac{1}{2}\left[\beta(\beta-1)s_m x(t)^{\beta-2} - \alpha(\alpha-1)r_M x(t)^{\alpha-2}\right] \\
&= \frac{1}{2}x(t)^{\alpha-2}[\alpha(1-\alpha)r_M - \beta(1-\beta)s_m x(t)^{\beta-\alpha}] \\
&\leq \frac{1}{2}x(t)^{\alpha-2}\left[\alpha(1-\alpha)r_M - \beta(1-\beta)s_m\sigma_2^{\beta-\alpha}\right] \\
&= \frac{1}{2}x(t)^{\alpha-2}\left[\alpha(1-\alpha)r_M - \beta(1-\beta)s_m\left(\frac{r_m}{s_M}\right)\right] \\
&= \frac{1}{2}x(t)^{\alpha-2}r_M\left[\alpha(1-\alpha) - \beta(1-\beta)\frac{1}{\Delta}\right] \\
&< 0.
\end{aligned}
$$

Finally, taking into account that $b(t) < 0$, (3.4), (3.10), (S) and reasoning as in the proof of Lemma 3.1 we have

$$\begin{aligned}
\tilde{b}(t) &= -b(t)\\
&\geq \frac{1}{2}\left[\beta(1-\beta)s_m x(t)^{\beta-2} - \alpha(1-\alpha)r_M x(t)^{\alpha-2}\right]\\
&\geq \frac{1}{2}\Big[\beta(1-\beta)s_m \sigma_1^{\beta-2} - \alpha(1-\alpha)r_M \sigma_1^{\alpha-2}\Big]\\
&= \frac{1}{2}\sigma_1^{\alpha-2}\Big[\beta(1-\beta)s_m \sigma_1^{\beta-\alpha} - \alpha(1-\alpha)r_M\Big]\\
&= \frac{1}{2}\left(\frac{r_M}{s_m}\right)^{\gamma(\alpha-2)}\Big[\beta(1-\beta)s_m\left(\frac{r_M}{s_m}\right) - \alpha(1-\alpha)r_M\Big]\\
&= \frac{1}{2}\left(\frac{r_M}{s_m}\right)^{\gamma(\alpha-2)} r_M\Big[\beta(1-\beta) - \alpha(1-\alpha)\Big],
\end{aligned}$$

and

$$\begin{aligned}
\tilde{b}(t) &\leq \frac{1}{2}\left[\beta(1-\beta)s_M x(t)^{\beta-2} - \alpha(1-\alpha)r_m x(t)^{\alpha-2}\right]\\
&\leq \frac{1}{2}\Big[\beta(1-\beta)s_M \sigma_2^{\beta-2} - \alpha(1-\alpha)r_m \sigma_2^{\alpha-2}\Big]\\
&= \frac{1}{2}\sigma_2^{\alpha-2}\Big[\beta(1-\beta)s_M \sigma_2^{\beta-\alpha} - \alpha(1-\alpha)r_m\Big]\\
&= \frac{1}{2}\left(\frac{r_m}{s_M}\right)^{\gamma(\alpha-2)}\Big[\beta(1-\beta)s_M\left(\frac{r_m}{s_M}\right) - \alpha(1-\alpha)r_m\Big]\\
&= \frac{1}{2}\left(\frac{r_m}{s_M}\right)^{\gamma(\alpha-2)} r_m\Big[\beta(1-\beta) - \alpha(1-\alpha)\Big].
\end{aligned}$$

□

The following lemma, which is easy to check in a similar way to Lemma 3.1, provides us with computable bounds for c_M.

Lemma 3.7. *Assume that r, s are positive T-periodic continuous functions and that x is a solution of equation (1.3) satisfying* (3.4) *(for instance if the conditions of Theorem 3.2 are fulfilled), that is*

$$\sigma_2 = \left(\frac{r_m}{s_M}\right)^{\gamma} \leq x(t) \leq \left(\frac{r_M}{s_m}\right)^{\gamma} = \sigma_1.$$

If (S) is satisfied, then the following claims hold:

(i) *For all $t \in \mathbb{R}$ and $x > 0$,*

$$f_{xxx}(t,x) \leq g_1(x) := \frac{1}{6}\left(\beta(1-\beta)(2-\beta)s_M x^{\beta-3} - \alpha(1-\alpha)(2-\alpha)r_m x^{\alpha-3}\right),$$

and the only positive solution of $g_1'(x) = 0$ is

$$x_1 = \left(\frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)r_m}{\beta(1-\beta)(2-\beta)(3-\beta)s_M}\right)\gamma.$$

Moreover, g_1 is increasing on $(0, x_1)$ and decreasing on $(x_1, +\infty)$.

(ii) *If* $\dfrac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{\beta(1-\beta)(2-\beta)(3-\beta)} \leq 1$ *then*

$$c_M \leq \max_{x\in[\sigma_2,\sigma_1]} g_1(x) = g_1(\sigma_2) = \frac{1}{6}((1-\beta)(2-\beta)\beta-(1-\alpha)(2-\alpha)\alpha)r_m \left(\frac{r_m}{s_M}\right)^{\gamma(\alpha-3)}.$$

(iii) *If* $1 < \dfrac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{\beta(1-\beta)(2-\beta)(3-\beta)} < \Delta$ *then*

$$c_M \leq \max_{x\in[\sigma_2,\sigma_1]} g_1(x) = g_1(x_1) = \frac{1}{6}\left(\frac{(\alpha-3)(\alpha-2)(\alpha-1)\alpha}{\beta-3} - (\alpha-2)(\alpha-1)\alpha\right) r_m\, x_1^{\alpha-3}.$$

(iv) *If* $\dfrac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{\beta(1-\beta)(2-\beta)(3-\beta)} \geq \Delta$ *then*

$$c_M \leq \max_{x\in[\sigma_2,\sigma_1]} g_1(x) = g_1(\sigma_1) = \frac{1}{6}r_m((1-\beta)(2-\beta)\beta\Delta-(1-\alpha)(2-\alpha)\alpha)\left(\frac{r_M}{s_m}\right)^{\gamma(\alpha-3)}.$$

The computable bounds for c_m are given in the following lemma.

Lemma 3.8. *Assume that r, s are positive T-periodic continuous functions and that x is a solution of equation (1.3) satisfying* (3.4). *If (S) holds, then:*

(i) *For all $t \in \mathbb{R}$ and $x > 0$,*

$$f_{xxx}(t,x) \geq g_2(x) := \frac{1}{6}\left(\beta(1-\beta)(2-\beta)s_m x^{\beta-3} - \alpha(1-\alpha)(2-\alpha)r_M x^{\alpha-3}\right),$$

and the only positive solution of $g_2'(x) = 0$ is

$$x_2 = \left(\frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)r_M}{\beta(1-\beta)(2-\beta)(3-\beta)s_m}\right)^{\gamma}.$$

Moreover, g_2 is increasing on $(0, x_2)$ and decreasing on $(x_2, +\infty)$.

(ii) *If* $\dfrac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{\beta(1-\beta)(2-\beta)(3-\beta)} \geq 1$ *then*

$$c_m \geq \min_{x\in[\sigma_2,\sigma_1]} g_2(x) = g_2(\sigma_2) = \frac{1}{6}((1-\beta)(2-\beta)\beta\frac{1}{\Delta}-(1-\alpha)(2-\alpha)\alpha)r_M \left(\frac{r_m}{s_M}\right)^{(\alpha-3)\gamma}.$$

(iii) *If* $1 > \dfrac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{\beta(1-\beta)(2-\beta)(3-\beta)} > \dfrac{1}{\Delta}$ *then*

$$c_m \geq \min_{x\in[\sigma_2,\sigma_1]} g_2(x) = \min\{g_2(\sigma_2), g_2(\sigma_1)\} =$$

$$\frac{1}{6}r_M \min\left\{\left(\frac{r_m}{s_M}\right)^{(\alpha-3)\gamma}((1-\beta)(2-\beta)\beta\frac{1}{\Delta} - (1-\alpha)(2-\alpha)\alpha),\right.$$
$$\left.\left(\frac{r_M}{s_m}\right)^{(\alpha-3)\gamma}((1-\beta)(2-\beta)\beta - (1-\alpha)(2-\alpha)\alpha)\right\}.$$

(iv) *If* $\dfrac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{\beta(1-\beta)(2-\beta)(3-\beta)} \leq \dfrac{1}{\Delta}$ *then*

$$c_m \geq \min_{x\in[\sigma_2,\sigma_1]} g_2(x) = g_2(\sigma_1) = \frac{1}{6}r_M((1-\beta)(2-\beta)\beta-(1-\alpha)(2-\alpha)\alpha)\left(\frac{r_M}{s_m}\right)^{(\alpha-3)\gamma}.$$

Now, we present our first stability result for equation (1.3).

Theorem 3.9. *Let us assume that r, s are positive T-periodic continuous functions and*

$$\Delta < \frac{\beta(1-\beta)(2-\beta)}{\alpha(1-\alpha)(2-\alpha)}, \tag{3.12}$$

$$\frac{r_m}{\gamma}\left(\frac{s_M}{r_m}\right)^{\gamma(1-\alpha)} < \left(\frac{\pi}{2T}\right)^2, \tag{3.13}$$

$$\Delta \leq \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{\beta(1-\beta)(2-\beta)(3-\beta)}, \tag{3.14}$$

and

$$\frac{5}{3}\gamma\Delta^{\frac{5}{2}\gamma(\alpha-1)}\Delta_r^{\frac{7}{2}}(\beta(1-\beta)-\alpha(1-\alpha))^2 > (1-\beta)(2-\beta)\beta\Delta-(1-\alpha)(2-\alpha)\alpha. \tag{3.15}$$

Then there exists a stable T-periodic solution of (1.3) *satisfying* (3.4).

Proof. Since (3.12) and (3.13) imply condition (C1) of Theorem 3.2, the existence of a solution of (1.3) satisfying (3.4) follows. The stability of such solution is a consequence of Lemma 2.5, (i), (ii) and (iii), taking into account the estimates obtained in Lemmas 3.6 and 3.7. □

Remark 3.10. *Condition* (3.14) *allows us to use part (iv) of Lemma 3.7 in order to get condition* (3.15). *Assuming the hypotheses of parts (ii) or (iii) of Lemma 3.7 instead of* (3.14) *will lead to alternate versions of* (3.15).

Our next stability result, analogous to Theorem 3.9, follows from (i), (ii) and (iii') of Lemma 2.5 and the estimates from Lemmas 3.6 and 3.8.

Theorem 3.11. *Let us assume that r, s are positive T-periodic continuous functions, and* (3.12)-(3.13) *hold. Moreover, if*

$$1 \leq \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{\beta(1-\beta)(2-\beta)(3-\beta)}, \tag{3.16}$$

and

$$\frac{5}{3}\gamma(\beta(1-\beta)-\alpha(1-\alpha))^2 < \Delta^{\frac{5}{2}\gamma(\alpha-1)}\Delta_r^{\frac{7}{2}}((1-\beta)(2-\beta)\beta\frac{1}{\Delta}-(1-\alpha)(2-\alpha)\alpha), \tag{3.17}$$

then there exists a stable T-periodic solution of (1.3) *satisfying* (3.4).

Remark 3.12. *Condition* (3.16) *allows us to use part (ii) of Lemma 3.8 in order to get condition* (3.17). *Assuming the hypotheses of parts (iii) or (iv) of Lemma 3.8 instead of* (3.16) *will lead to alternate versions of* (3.17).

In the remainder of this section we provide a stability criterium for (1.4).

Theorem 3.13. *Assume that r, s are T-periodic continuous functions, and $\overline{r}$ and s are positive. Then the T-periodic solution $x(t,\varepsilon)$ of* (1.4) *obtained in Theorem* 3.4 *is stable if $\varepsilon > 0$ is small enough and the following conditions are satisfied*

$$2\alpha^2 + 2\beta^2 + 7\alpha\beta - \alpha - \beta - 1 \neq 0, \tag{3.18}$$

$$\frac{\alpha r_M}{\beta s_m} \leq \omega := \frac{\overline{r}}{\overline{s}}. \tag{3.19}$$

Proof. To prove our theorem it is enough to show that the first twist coefficient μ given by (2.2) is different from zero. Observe that the third-order approximation of (1.4) is

$$\ddot{y} + a(t)y + b(t)y^2 + c(t)y^3 + o(y^3) = 0,$$

where

$$a(t) = \varepsilon\beta s(t)x(t)^{\beta-1} - \alpha r(t)x(t)^{\alpha-1}, \tag{3.20}$$

$$b(t) = \frac{1}{2}\left[\varepsilon\beta(\beta-1)s(t)x(t)^{\beta-2} - \alpha(\alpha-1)r(t)x(t)^{\alpha-2}\right], \tag{3.21}$$

and

$$c(t) = \frac{1}{6}\left[\varepsilon\beta(\beta-1)(\beta-2)s(t)x(t)^{\beta-3} - \alpha(\alpha-1)(\alpha-2)r(t)x(t)^{\alpha-3}\right]. \tag{3.22}$$

By inserting the limit (3.6) into (3.20)-(3.22), we have

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\gamma(\alpha-1)}a(t) = \beta s(t)\omega^{\gamma(\beta-1)} - \alpha r(t)\omega^{\gamma(\alpha-1)}, \tag{3.23}$$

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\gamma(\alpha-2)}b(t) = \frac{1}{2}\left[\beta(\beta-1)s(t)\omega^{\gamma(\beta-2)} - \alpha(\alpha-1)r(t)\omega^{\gamma(\alpha-2)}\right], \tag{3.24}$$

and

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\gamma(\alpha-3)}c(t) = \frac{1}{6}\left[\beta(\beta-1)(\beta-2)s(t)\omega^{\gamma(\beta-3)} - \alpha(\alpha-1)(\alpha-2)r(t)\omega^{\gamma(\alpha-3)}\right]. \tag{3.25}$$

Using condition (3.19) in (3.23), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \varepsilon^{\gamma(\alpha-1)}a(t) &= \beta s(t)\omega^{\gamma(\beta-1)} - \alpha r(t)\omega^{\gamma(\alpha-1)} \\ &\geq \beta s_m\omega^{\gamma(\beta-1)} - \alpha r_M\omega^{\gamma(\alpha-1)} \\ &= \omega^{\gamma(\alpha-1)}(\beta s_m\omega - \alpha r_M) \geq 0,\end{aligned}$$

which implies that $a(t) \geq 0$ if $\varepsilon > 0$ is small enough. Moreover,

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\gamma(\alpha-1)}\bar{a} = (\beta-\alpha)\frac{\bar{r}^{\gamma(\beta-1)}}{\bar{s}^{\gamma(\alpha-1)}},$$

which implies that $\bar{a} > 0$ if $\varepsilon > 0$ is small enough and that $\lim_{\varepsilon\to 0^+} \bar{a} = 0^+$.

Application of Lemma 2.6 gives

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\frac{\gamma(\alpha-1)}{2}}\theta = T\sqrt{\beta-\alpha}\sqrt{\frac{\bar{r}^{\gamma(\beta-1)}}{\bar{s}^{\gamma(\alpha-1)}}}, \tag{3.26}$$

and

$$\lim_{\varepsilon\to 0^+} \frac{R(t)}{\varepsilon^{\frac{\gamma(\alpha-1)}{4}}} = \frac{1}{\sqrt[4]{\beta-\alpha}}\sqrt[4]{\frac{\bar{s}^{\gamma(\alpha-1)}}{\bar{r}^{\gamma(\beta-1)}}} \tag{3.27}$$

for $\varepsilon > 0$ small enough.

It follows from [13, Lemma 3.6] that (2.1) is elliptic and 4-elementary if $\varepsilon > 0$ is small enough. Moreover, it is proved in [21] that the kernel χ_θ in (2.2) is symmetric with respect to the line $x = \theta/2$, χ_θ is strictly increasing on $[0, \theta/2]$ and strictly decreasing on $[\theta/2, \theta]$. Therefore,

$$\begin{aligned} \min_{x\in[0,\theta]} \chi_\theta(x) = \chi_\theta(0) &= \frac{3\cos(\theta/2)}{16\sin(\theta/2)} + \frac{\cos(3\theta/2)}{16\sin(3\theta/2)} \\ &= \frac{5}{8\sin(3\theta/2)} \frac{(1+4\cos\theta)\cos(\theta/2)}{5}, \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} \max_{x\in[0,\theta]} \chi_\theta(x) = \chi_\theta(\theta/2) &= \frac{3}{16\sin(\theta/2)} + \frac{1}{16\sin(3\theta/2)} \\ &= \frac{5}{8\sin(3\theta/2)} \frac{10-12\sin^2(\theta/2)}{10}. \end{aligned} \tag{3.29}$$

We deduce from (3.28)-(3.29) that for $\varepsilon > 0$ small enough we get

$$\chi_\theta(x) = \frac{5}{12\theta}\left(1 + O(\theta^2)\right) = \frac{5}{12}(T\sqrt{\bar{a}})^{-1} + O(\bar{a}),$$

in which we have used Lemma 2.6. This gives

$$\lim_{\varepsilon\to 0^+} [\varepsilon^{\frac{1}{2}\gamma(1-\alpha)}\chi_\theta(|\varphi(t)-\varphi(\tau)|)] = \frac{5}{12T}\frac{1}{\sqrt{\beta-\alpha}}\sqrt{\frac{\bar{r}^{\gamma(1-\beta)}}{\bar{s}^{\gamma(1-\alpha)}}}.$$

Let

$$\mu_1 = \iint_{[0,T]^2} b(t)b(\tau)R^3(t)R^3(\tau)\chi_\theta(|\varphi(t)-\varphi(\tau)|)dtd\tau,$$

and

$$\mu_2 = \int_0^T c(t)R^4(t)dt.$$

Using (3.23)-(3.27), we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \frac{\mu_2}{\varepsilon^{2\gamma}} &= \lim_{\varepsilon\to 0^+} \int_0^T \varepsilon^{\gamma(\alpha-3)} c(t) \cdot \frac{R^4(t)}{\varepsilon^{\gamma(\alpha-1)}} dt \\ &= \int_0^T \frac{1}{6}\left[\beta(\beta-1)(\beta-2)s(t)\omega^{\gamma(\beta-3)} - \alpha(\alpha-1)(\alpha-2)r(t)\omega^{\gamma(\alpha-3)}\right] \\ &\quad \cdot \frac{1}{\beta-\alpha}\frac{\bar{s}^{\gamma(\alpha-1)}}{\bar{r}^{\gamma(\beta-1)}} dt \\ &= \frac{T\bar{s}^{\gamma(\alpha-1)}}{6(\beta-\alpha)\bar{r}^{\gamma(\beta-1)}}\left[\beta(\beta-1)(\beta-2)\omega^{\gamma(\beta-3)}\bar{s} - \alpha(\alpha-1)(\alpha-2)\omega^{\gamma(\alpha-3)}\bar{r}\right] \\ &= \frac{T}{6(\beta-\alpha)\omega^{2\gamma}}\left[\beta(\beta-1)(\beta-2) - \alpha(\alpha-1)(\alpha-2)\right] \\ &= \frac{T}{6\omega^{2\gamma}}(\alpha^2 + \alpha\beta + \beta^2 - 3\alpha - 3\beta + 2). \end{aligned}$$

Clearly,

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \frac{\mu_1}{\varepsilon^{2\gamma}} = \lim_{\varepsilon\to 0^+} \iint_{[0,T]^2} & \varepsilon^{\gamma(\alpha-2)} b(t) \cdot \varepsilon^{\gamma(\alpha-2)} b(\tau) \cdot \left[\frac{R^3(t)}{\varepsilon^{\frac{3\gamma(\alpha-1)}{4}}}\right] \\ & \cdot \left[\frac{R^3(\tau)}{\varepsilon^{\frac{3\gamma(\alpha-1)}{4}}}\right] \cdot \left[\varepsilon^{\frac{1}{2}\gamma(1-\alpha)}\chi_\theta(|\varphi(t)-\varphi(\tau)|)\right] dtd\tau. \end{aligned}$$

Therefore, in the same manner we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\frac{\mu_1}{\varepsilon^{2\gamma}} &= \iint_{[0,T]^2}\frac{1}{4}\left[\beta(\beta-1)\omega^{\gamma(\beta-2)}s(t)-\alpha(\alpha-1)\omega^{\gamma(\alpha-2)}r(t)\right]\\
&\quad\cdot\left[\beta(\beta-1)\omega^{\gamma(\beta-2)}s(\tau)-\alpha(\alpha-1)\omega^{\gamma(\alpha-2)}r(\tau)\right]\\
&\quad\times\frac{1}{(\beta-\alpha)^{3/2}}\left[\frac{\bar{s}^{\gamma(\alpha-1)}}{\bar{r}^{\gamma(\beta-1)}}\right]^{3/2}\cdot\frac{5}{12T}\frac{1}{\sqrt{\beta-\alpha}}\sqrt{\frac{\bar{r}^{\gamma(1-\beta)}}{\bar{s}^{\gamma(1-\alpha)}}}dtd\tau\\
&= \frac{5}{48T}\frac{1}{(\beta-\alpha)^2}\frac{\bar{s}^{2\gamma(\alpha-1)}}{\bar{r}^{2\gamma(\beta-1)}}\iint_{[0,T]^2}\left[\beta(\beta-1)\omega^{\gamma(\beta-2)}s(t)-\alpha(\alpha-1)\omega^{\gamma(\alpha-2)}r(t)\right]\\
&\quad\times\left[\beta(\beta-1)\omega^{\gamma(\beta-2)}s(\tau)-\alpha(\alpha-1)\omega^{\gamma(\alpha-2)}r(\tau)\right]dtd\tau\\
&= \frac{5}{48T}\frac{1}{(\beta-\alpha)^2}\frac{\bar{s}^{2\gamma(\alpha-1)}}{\bar{r}^{2\gamma(\beta-1)}}\left[\beta(\beta-1)T\bar{s}\omega^{\gamma(\beta-2)}-\alpha(\alpha-1)T\bar{r}\omega^{\gamma(\alpha-2)}\right]^2\\
&= \frac{5T}{48\omega^{2\gamma}}(\alpha+\beta-1)^2.
\end{aligned}$$

Hence, by (2.2)

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\frac{\mu}{\varepsilon^{2\gamma}} &= \lim_{\varepsilon\to 0^+}\frac{\mu_1-\frac{3}{8}\mu_2}{\varepsilon^{2\gamma}}\\
&= \frac{T}{\omega^{2\gamma}}\left[\frac{5(\alpha+\beta-1)^2}{48}-\frac{\alpha^2+\alpha\beta+\beta^2-3\alpha-3\beta+2}{16}\right]\\
&= \frac{T}{48\omega^{2\gamma}}(2\alpha^2+2\beta^2+7\alpha\beta-\alpha-\beta-1).
\end{aligned}$$

Condition (3.18) implies that

$$\lim_{\varepsilon\to 0^+}\frac{\mu}{\varepsilon^{2\gamma}}\neq 0,$$

which means that the twist coefficient μ is non-zero when ε is small enough. This finishes the proof. □

Corollary 3.14. *Suppose that r and s are T-periodic continuous functions, $\bar{r}$ and s are positive, α, β satisfy (3.18)-(3.19). Then the equation*

$$\ddot{x} = \lambda r(t)x^\alpha - s(t)x^\beta \tag{3.30}$$

has a stable T-periodic solution $x(t,\lambda)$ if λ is large enough.

Proof. If we introduce the variable

$$x = \lambda^{\frac{1}{1-\alpha}}y,$$

then (3.30) is changed to the equation

$$\ddot{y} = r(t)y^\alpha - \varepsilon s(t)y^\beta,$$

where

$$\varepsilon = \lambda^{\frac{\beta-1}{1-\alpha}}.$$

Note that $\varepsilon\to 0$ if and only if $\lambda\to+\infty$. So Corollary 3.14 holds from Theorem 3.13. □

4. Application to the Liebau model

In our next result we give explicit conditions in terms of the data in order to obtain the existence, localization, and stability of positive solutions of (1.6).

Theorem 4.1. *Let us assume $e_m > 0$. The following conclusions hold:*
(I) (Existence). *If*

$$c^2[be_M - (b-1)e_m] < \left(\frac{\pi}{T}\right)^2 e_m^2, \tag{4.1}$$

is satisfied, then there exists a T-periodic solution u of (1.6) such that

$$\frac{e_m}{c} \le u(t) \le \frac{e_M}{c} \quad \text{for all } t \in [0,T].$$

(II) (Stability). *If*

$$\Delta_e < \frac{b(b+2)}{2(b-1)(b+3)}, \tag{4.2}$$

$$\frac{c^2}{e_m} < \left(\frac{\pi}{2T}\right)^2, \tag{4.3}$$

$$\Delta_e \le \frac{4(b-1)(b+3)}{b(2b+3)}, \tag{4.4}$$

and

$$3b(b+2)\Delta_e^{5/2} - 6(b-1)(b+3)\Delta_e^{3/2} < 5(b-2)^2, \tag{4.5}$$

are satisfied, then there exists a stable T-periodic solution u of (1.6) such that

$$\frac{e_m}{c} \le u(t) \le \frac{e_M}{c} \quad \text{for all } t \in [0,T].$$

(III) (Stability). *If* (4.2), (4.3),

$$b \ge \frac{3}{2}, \tag{4.6}$$

and

$$5(b-2)^2\Delta_e^{5/2} + 6(b-1)(b+3)\Delta_e < 3b(b+2), \tag{4.7}$$

are satisfied, then there exists a stable T-periodic solution u of (1.6) such that

$$\frac{e_m}{c} \le u(t) \le \frac{e_M}{c} \quad \text{for all } t \in [0,T].$$

Proof. Since $r(t) = (b+1)e(t)$ and $s(t) = (b+1)c$, the existence follows from Corollary 3.3 taking $\alpha = (b-1)\kappa$ and $\beta = b\kappa$. On the other hand, the stability results *(II)* and *(III)* are a direct consequence of Theorems 3.9 and 3.11, respectively. □

Remark 4.2. *Note that in Theorem 4.1 the stability conditions in parts (II) and (III) also imply existence.*

On the other hand, since $\Delta_e \ge 1$, a necessary condition in order to apply either part (II) or part (III) of Theorem 4.1 is

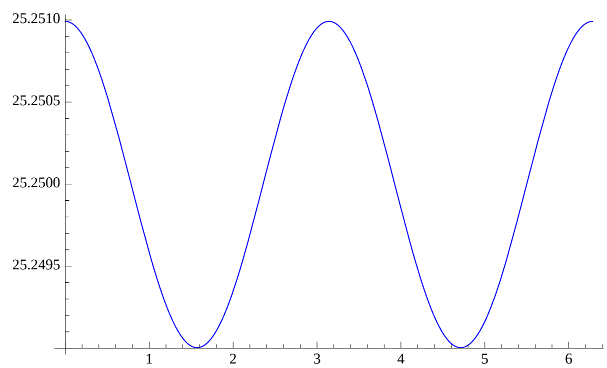
$$1.5 \le b < \sqrt{7} - 1 \approx 1.64575.$$

Example 4.3. *The equation*

$$\ddot{u} = \frac{1}{u}(0.2\sin^2 t + 10 - b\dot{u}^2) - c \tag{4.8}$$

has at least one stable 2π-periodic positive solution provided that $1.5 \le b < 1.56138$ and $0 < c < 0.790569$.

Proof. Let us define $e(t) = 0.2\sin^2 t + 10$ and $T = 2\pi$. Then we can compute that $e_M = 10.2, e_m = 10$ and $\Delta_e = 1.02$. All conditions of Theorem 4.1 *(III)* are satisfied if $1.5 \le b < 1.56138$ and $0 < c < 0.790569$, thus we get that (4.8) has at least one stable 2π-periodic positive solution.



2π-periodic solution of equation (4.8) with $b = 1.55$ and $c = 0.4$.

□

Finally, by considering c as a small parameter in (1.6), we obtain the following result about the existence and stability of periodic solutions.

Theorem 4.4. *Let us assume $\bar{e} > 0$. Then the following conclusions hold:*
(I) (Existence). *For c small enough, there exists at least one T-periodic solution $u(t, c)$ of (1.6) and moreover*

$$\lim_{c \to 0} c\, u(t, c) = \bar{e}, \quad uniformly \ in \ t.$$

(II) (Stability). *The periodic solution found in part (I) is stable provided that $b \neq \frac{7+\sqrt{33}}{8}$ and $(b-1)e_M - \bar{e}b \le 0$.*

Proof. Since $r(t) = (b+1)e(t)$ and $s(t) = b+1$, condition $\bar{e} > 0$ implies that $\bar{r} > 0$. Now the existence follows by using Theorem 3.4. Concerning the stability, we apply Theorem 3.13. □

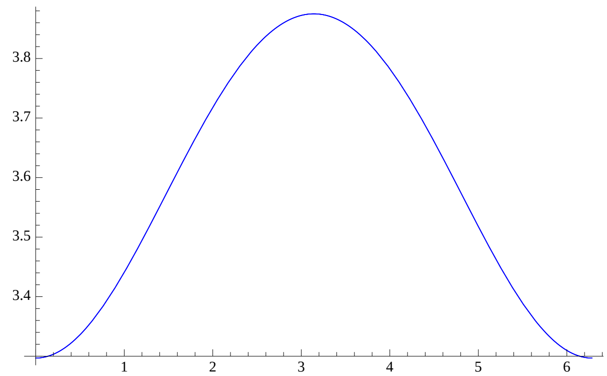
Remark 4.5. *We point out that $\bar{e} > 0$ assumed in Theorem 4.4 is a necessary condition for the existence of a periodic positive solution of (1.6), see* [5]. *It is an open problem explicitly stated in* [19, Chapter 8] *whether $\bar{e} > 0$ is also a sufficient condition for existence. So, Theorem 4.4 can be viewed as a partial answer to this open problem which in addition provides stability information.*

Using Theorem 4.4, we can prove the following result with a sign changing nonlinearity.

Example 4.6. *The equation*

$$\ddot{u} = \frac{1}{u}(\cos t + \frac{1}{2} - b\dot{u}^2) - c \tag{4.9}$$

has at least one stable 2π*-periodic positive solution provided that* $1 < b \leq \frac{3}{2}$ *and* c *is small enough.*



2π-periodic positive solution of equation (4.9) with $b = 3/2$ and $c = 0.133333$.

Remark 4.7. *Note that Theorem 4.1 cannot be applied to (4.9) in order to guarantee the existence and stability of* 2π*-periodic solution since* $e(t) = \cos t + \frac{1}{2}$ *takes negative values.*

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References

1. J. Chu, M. Li, Twist periodic solutions of second order singular differential equations, J. Math. Anal. Appl. 355 (2009), 830-838.
2. J. Chu, P.J. Torres, F. Wang, Radial stability of periodic solutions of the Gylden-Meshcherskii-type problem, Discrete Contin. Dyn. Syst. 35 (2015), 1921-1932.
3. J. Chu, P.J. Torres, F. Wang, Twist periodic solutions for differential equations with a combined attractive-repulsive singularity, J. Math. Anal. Appl. 437 (2016), 1070-1083.
4. J. Chu, M. Zhang, Rotation number and Lyapunov stability of elliptic periodic solutions, Discrete Contin. Dyn. Syst. 21 (2008), 1071-1094.
5. J.A. Cid, G. Propst, M. Tvrdý, On the pumping effect in a pipe/tank flow configuration with friction, Phys. D 273-274 (2014), 28-33.
6. J.A. Cid, G. Infante, M. Tvrdý, M. Zima, A topological approach to periodic oscillations related to the Liebau phenomenon, J. Math. Anal. Appl. 423 (2015), 1546-1556.
7. J.A. Cid, G. Infante, M. Tvrdý, M. Zima, New results for the Liebau phenomenon via fixed point index, Nonlinear Anal. RWA 35 (2017), 457-469.
8. C. De Coster, P. Habets, Two-point boundary value problems: lower and upper solutions, Mathematics in Science and Engineering, 205, Elsevier B. V., Amsterdam, 2006.
9. E.N. Dancer, R. Ortega, The index of Lyapunov stable fixed points in two dimensions, J. Dynam. Differential Equations 6 (1994), 631-637.
10. G. Feltrin, F. Zanolin, Multiple positive solutions for a superlinear problem: A topological approach, J. Differential Equations 259 (2015), 925-963.

11. J.K. Hale, Ordinary differential equations, Second Edition, Wiley-Interscience, New York, 1980.
12. F. Liao, Periodic solutions of Liebau-type differential equations, Appl. Math. Lett. 69 (2017), 8-14.
13. J. Lei, X. Li, P. Yan, M. Zhang, Twist character of the least amplitude periodic solution of the forced pendulum, SIAM J. Math. Anal. 35 (2003), 844-867.
14. G. Liebau, Über ein ventilloses Pumpprinzip, Naturwissenschaften 41 (1954), 327.
15. R. Ortega, Periodic solution of a Newtonian equation: stability by the third approximation, J. Differential Equations 128 (1996), 491-518.
16. R. Ortega, G. Verzini, A variational method for the existence of bounded solutions of a sublinear forced oscillator, Proc. London Math. Soc. 88 (2004), 775-795.
17. G. Propst, Pumping effects in models of periodically forced flow configurations, Phys. D 217 (2006), 193–201.
18. C. Siegel, J. Moser, Lectures on Celestial Mechanics, Springer-Verlag, Berlin, 1971.
19. P.J. Torres, Mathematical Models with Singularities - A Zoo of Singular Creatures, Atlantis Press, 2015.
20. P.J. Torres, M. Zhang, Twist periodic solutions of repulsive singular equations, Nonlinear Anal. 56 (2004), 591-599.
21. M. Zhang, The best bound on the rotations in the stability of periodic solutions of a Newtonian equation, J. London Math. Soc. 67 (2003), 137-148.

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