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Sharp conditions for the existence of solutions of second-order autonomous differential equations

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Abstract. We prove general existence results for

$$x'' = f(x)g(x'), \quad x(0) = x_0,\, x'(0) = x_1,$$

where f and g need not be continuous or monotone. Moreover neither f nor g need be bounded around, respectively, x_0 and x_1, thus allowing singularities in the equation. Several other basic topics such as uniqueness, continuation, extremality and periodicity are studied in our general framework.

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1. Introduction

This paper is concerned with the derivation of sharp sufficient conditions for the existence of Carathéodory solutions for the initial value problem

$$x'' = f(x)g(x'), \quad x(0) = x_0, \quad x'(0) = x_1. \tag{1.1}$$

We recall that a Carathéodory solution of (1.1) is a locally absolutely continuous function which satisfies the initial conditions and fulfills the differential equation almost everywhere in its domain.

In order to motivate the type of conditions that we are going to impose over f and g we recall some previously published results.

It is proven in [6, theorem 3.1] that problem

$$x'' = f(x), \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1.2}$$

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has a nonconstant solution $x : I \to \mathbb{R}$ with $x(I) = J$ if and only if

(P) $f \in L^1_{loc}(J)$, $x_1^2 + 2\int_{x_0}^{y} f(s)ds > 0$ for almost all $y \in J$ and

$$\frac{\max\{1, |f|\}}{\sqrt{x_1^2 + 2\int_{x_0}^{\cdot} f(s)ds}} \in L^1_{loc}(J).$$

Moreover, if condition (P) holds then (1.2) has a strictly monotone solution x implicitly given by the expression

$$\int_{x_0}^{x(t)} \frac{dr}{\sqrt{x_1^2 + 2\int_{x_0}^{r} f(s)ds}} = \operatorname{sgn}(x_1)t \quad \text{for all } t \in \operatorname{sgn}(x_1)\tau(J),$$

where $\operatorname{sgn}(z) = z/|z|$ for $z \neq 0$, $\operatorname{sgn}(0) \in \{-1, 1\}$, and

$$\tau : y \in J \longmapsto \tau(y) := \int_{x_0}^{y} \frac{dr}{\sqrt{x_1^2 + 2\int_{x_0}^{r} f(s)ds}}.$$

In this paper we fix a class of proper functions g, which includes the model function $g = 1$, and we give necessary and sufficient conditions over f for the existence of solutions of (1.1), extending in this way [6, theorem 3.1]. We remark that useful necessary conditions of existence over both f and g seem hard to find, as (1.1) may have nontrivial solutions for pairs of nonmeasurable f and g. As an example note that the affine function $x(t) = x_1 t + x_0$, $t \in \mathbb{R}$, solves (1.1) if $g(x_1) = 0$, without further conditions over f and g.

Our main arguments lean on time maps, which play an important role in the analysis of second-order equations, see [5, 8].

We finish this introduction with a description of the organization of the present paper. In section 2 we study the case $f = 1$ and we deduce necessary and sufficient conditions for solving it. To the best of our knowledge, that result is new. In section 3 we introduce some preliminary results on change of variables in the Lebesgue integral which lead to new criteria for the absolute continuity of compositions of absolutely continuous functions. In section 4 we fix some conditions for g and we deduce necessary conditions on f for the existence of Carathéodory solutions for (1.1). It is worth to note that some of the arguments used in [6] for (1.1) with $g = 1$ are no longer valid in our more general setting. This forces us to employ more complicated arguments in terms of down-up and up-down functions, which are concepts defined in this section. In section 5 we show that the previous necessary conditions are sufficient too, thus closing our circular existence result; moreover we are able to establish even more general sufficient conditions for solving (1.1) by relaxing some of the assumptions on g. In section 6 we study uniqueness and non-uniqueness of solutions and in section 7 we consider continuation of solutions and we prove a result on periodic solutions.

2. Some reasonable conditions to impose on g

Note first that in the case $f = 1$ a function $x : I \to \mathbb{R}$ is a solution of

$$x''(t) = g(x'(t)) \text{ for a.a. } t, \quad x(0) = x_0, \quad x'(0) = x_1, \tag{2.1}$$

if and only if $y = x'$ is a solution of the first-order problem

$$y' = g(y) \text{ for a.a. } t, \quad y(0) = x_1, \tag{2.2}$$

whose solvability (in the Carathéodory sense) was completely studied by Binding in [1].

A very special feature of (2.2) is that solutions are monotone. We can summarize Binding's existence results as follows:

Theorem 2.1. *Problem (2.2) has the constant solution if and only if $g(x_1) = 0$.*

Moreover, the following conditions are pairwise equivalent for a nontrivial interval J that contains x_1:

1. *(2.2) has a nondecreasing (nonincreasing) solution with range J;*
2. *$g(y) > 0$ $(g(y) < 0)$ for a.a. $y \in J$ and $1/g \in \mathrm{L}^1_{loc}(J)$;*
3. *(2.2) has a solution with range J and positive (negative) derivative almost everywhere in its domain.*

As a consequence of theorem 2.1 we have a similar result concerning (2.1). Note that solutions of (2.1) are either concave or convex on their domains.

Corollary 2.2. *Problem (2.1) has the affine solution if and only if $g(x_1) = 0$.*

Moreover, the following conditions are pairwise equivalent for a nontrivial interval $\tilde{J}$ that contains x_1:

1. *(2.2) has a non-affine convex (concave) solution whose derivative has range $\tilde{J}$;*
2. *$g(y) > 0$ $(g(y) < 0)$ for a.a. $y \in \tilde{J}$ and $1/g \in \mathrm{L}^1_{loc}(\tilde{J})$;*
3. *(2.2) has a solution with range of derivative $\tilde{J}$ and positive (negative) second derivative almost everywhere in its domain.*

Remark 2.3. Solutions can be defined on one side of $t = 0$ or on both. Specifically, in the conditions of corollary (2.2) we can say that if for some $\varepsilon > 0$ we have $g(y) > 0$ a.e. on $(x_1, x_1 + \varepsilon)$ and $1/g \in \mathrm{L}^1(x_1, x_1 + \varepsilon)$ then we have a convex solution defined on the right of $t = 0$, and if $g(y) > 0$ a.e. on $(x_1 - \varepsilon, x_1)$ and $1/g \in \mathrm{L}^1(x_1 - \varepsilon, x_1)$ then we have a convex solution on the left of $t = 0$. The situation is similar for the remaining possibilities.

Corollary 2.2 gives us the weakest possible conditions for having solutions of (2.2), because they are necessary. In this sense those conditions are upper bounds for the generality we can expect when studying (1.1), and the same is true for conditions (P) on f.

3. Preliminary results

Part 1 of the next lemma was established as lemma 2.2, (1) in [6]. Note that, unlike the usual theorems on change of variable, assumptions are imposed on the composition $(f \circ x)x'$ instead of on f, which is important for our purposes in this paper. Specifically, this result is a key ingredient in our subsequent study of absolute continuity of compositions of absolutely continuous functions. Parts 2 and 3 of the lemma are respectively theorems 38.3 and 38.4 of [7].

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R} \cup \{-\infty, +\infty\}$ be measurable and let $x \in AC(I)$, $I = [t_1, t_2]$, be such that $a \le x(t) \le b$ for all $t \in I$.*

1. *If $(f \circ x)x' \in L^1(I)$ then $f \in L^1(x(I))$ and*
$$\int_{x(t_1)}^{x(t_2)} f(r)dr = \int_{t_1}^{t_2} f(x(t))x'(t)dt. \tag{3.1}$$
2. *If $f \in L^\infty(x(I))$ then $(f \circ x)x' \in L^1(I)$and*
$$\int_{x(t_1)}^{x(t_2)} f(r)dr = \int_{t_1}^{t_2} f(x(t))x'(t)dt. \tag{3.2}$$
3. *If $f \in L^1(x(I))$ and x is monotone then $(f \circ x)x' \in L^1(I)$and*
$$\int_{x(t_1)}^{x(t_2)} f(r)dr = \int_{t_1}^{t_2} f(x(t))x'(t)dt. \tag{3.3}$$

The preceding Lemma 3.1, part 1, allows us to establish a new result on absolute continuity of compositions of absolutely continuous functions. We remark that in general such compositions need not be absolutely continuous.

Lemma 3.2. *Let I and J be nontrivial compact intervals.*

Let $F : I \to \mathbb{R}$ be an absolutely continuous function on I with $F(I) \subset J$ and let $G : J \to \mathbb{R}$ be absolutely continuous on J. Then the following claims hold:

(i) *If $(G' \circ F)F' \in L^1(I)$ then $G \circ F$ is absolutely continuous on I and moreover*
$$(G \circ F)'(t) = G'(F(t))F'(t) \quad \text{for a. a. } t \in I.$$

(ii) *If $(G \circ F)' \in \mathrm{L}^1(I)$ and the set*
$$A = \{t \in I \,:\, G \text{ is not differentiable at } F(t)\}$$
has zero Lebesgue measure, then $G \circ F$ is absolutely continuous on I and moreover
$$(G \circ F)'(t) = G'(F(t))F'(t) \quad \text{for a. a. } t \in I.$$

Proof. (i) Let $t_0 \in I$. Since $(G' \circ F)F' \in L^1(I)$ we have from Lemma 3.1 that
$$\int_{t_0}^{t} G'(F(s))F'(s)ds = \int_{F(t_0)}^{F(t)} G'(s)ds \quad \text{for all } t \in I.$$

On the other hand G is absolutely continuous on J and then

$$\int_{F(t_0)}^{F(t)} G'(s)ds = G(F(t)) - G(F(t_0)) \quad \text{for all } t \in I.$$

Therefore

$$G(F(t)) - G(F(t_0)) = \int_{t_0}^{t} G'(F(s))F'(s)ds \quad \text{for all } t \in I,$$

and thus $G \circ F$ is absolutely continuous on I and

$$(G \circ F)'(t) = G'(F(t))F'(t) \quad \text{for a. a. } t \in I.$$

(ii) If $t \in I \setminus A$ is such that F is differentiable at t then the chain rule ensures that $G \circ F$ is differentiable at t and that

$$(G \circ F)'(t) = G'(F(t))F'(t).$$

Therefore $(G' \circ F)F' = (G \circ F)'$ almost everywhere and then $(G' \circ F)F' \in \mathrm{L}^1(I)$ so the assumptions of part (i) of the lemma are satisfied. □

As an immediate consequence of Lemma 3.2 (ii) we obtain the following result, which has a fundamental importance in [6] where it was proven in a completely different way.

Corollary 3.3. *Let $F : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$ such that $F(y) > 0$ for almost all $y \in [a, b]$.*

If $\frac{d}{dy}\sqrt{F} \in L^1(a, b)$ then $\sqrt{F}$ is absolutely continuous on $[a, b]$.

We also shall need the following result on absolute continuity of inverse functions. A simple proof for the next lemma, based on standard results left as exercises 5 (i) and 6 (b) in pages 332 and 333 of [9], is given in [2].

Lemma 3.4. *Let $I = [a, b]$ and $J = [c, d]$ be a pair of nontrivial intervals and let $F : I \to J$ be one-to-one and onto and absolutely continuous on I.*

If $F'(t) \neq 0$ for a.a. $t \in I$ then $F^{-1} : J \to I$ is absolutely continuous on J and

$$(F^{-1})'(r) = \frac{1}{F'(F^{-1}(r))} \quad \textit{for a.a. } r \in J.$$

4. Necessary conditions for existence of solutions

We start supposing that $g : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(g0) $g(y) > 0$ for all $y \in \mathbb{R}$.

(g1) $\dfrac{1}{g} \in \mathrm{L}^{\infty}_{loc}(\mathbb{R})$.

Now assuming (g0) and (g1) we present the necessary conditions on f for existence of solution. We define

$$G(y) := \int_0^y \frac{t}{g(t)} dt \quad \text{for all } y \in \mathbb{R}.$$

Remark 4.1. Under some additional conditions over g one could prove that the equation $x'' = f(x)g(x')$ would be equivalent to $(G(x'))' = f(x)x'$. Such transformations have already been noted and used by several authors, see [3, 4].

For the convenience of the reader we follow the presentation of [6].

Proposition 4.2. *Assume hypotheses (g0) and (g1).*

If $x : I \to \mathbb{R}$ is a solution of (1.1) then $f_{|x(I)}$ is Lebesgue-measurable.

Proof. Let $O \subset \mathbb{R} \cup \{-\infty, +\infty\}$ be open. Since $\frac{1}{g} \in \mathrm{L}^\infty_{loc}(x'(I))$ from lemma 3.1, part 2, we have that $\frac{x''}{g \circ x'} \in L^1_{loc}(I)$ and in particular is measurable on I. Then the set

$$\left(\frac{x''}{g \circ x'}\right)^{-1}(O) = x^{-1}(f^{-1}(O))$$

is a measurable subset of I. Therefore $x\left(\left(\frac{x''}{g(x')}\right)^{-1}(O)\right) = f^{-1}(O) \cap x(I)$ is measurable because x is absolutely continuous (see [9, exercise 6, page 333]). □

Conditions (g0) and (g1) are stronger than those imposed on g in corollary 2.2. Regrettably, we cannot establish proposition 4.2 under the conditions of corollary 2.2 for g. As an example note that $x(t) = t$, $t \in \mathbb{R}$, solves (1.1) with $x(0) = 0$ and $x'(0) = 1$ if $g(1) = 0$, independently of the choice of f.

Proposition 4.3. *Assume hypotheses (g0) and (g1).*

If $x : I \to \mathbb{R}$ is a solution of (1.1) then $f \in \mathrm{L}^1_{loc}(x(I))$ (in particular, $f \in \mathrm{L}^1(x(I))$ in case I is compact) and

$$G(x'(t_2)) - G(x'(t_1)) = \int_{x(t_1)}^{x(t_2)} f(r)dr \quad \text{for all } t_1, t_2 \in I. \tag{4.1}$$

Proof. For a.a. $t \in I$ we have

$$\frac{x''(t)}{g(x'(t))} x'(t) = f(x(t))x'(t),$$

and hence $(f \circ x)x' \in L^1_{loc}(I)$ because x' is continuous on I and $\frac{x''}{g(x')} \in L^1_{loc}(I)$ (as we have shown in the previous proof). In particular, $(f \circ x)x' \in L^1([t_1, t_2])$ for every $t_1, t_2 \in I$, $t_1 < t_2$. Moreover, by proposition 4.2, we know that f :

$x(I) \to \mathbb{R} \cup \{-\infty, +\infty\}$ is measurable, so we can apply the part 1 of lemma 3.1 and conclude that $f \in L^1(x([t_1, t_2]))$ and that

$$\int_{t_1}^{t_2} \frac{x''(s)}{g(x'(s))} x'(s) ds = \int_{x(t_1)}^{x(t_2)} f(r) dr.$$

Finally since $\dfrac{x}{g(x)} \in L^\infty_{loc}(\mathbb{R})$ applying lemma 3.1, part 2, to the left-hand side in the previous relation we obtain

$$\int_{t_1}^{t_2} \frac{x''(s)}{g(x'(s))} x'(s) ds = \int_{x'(t_1)}^{x'(t_2)} \frac{r}{g(r)} dr = G(x'(t_1)) - G(x'(t_2)),$$

and (4.1) holds. □

The following result follows immediately from (4.1) with $t_1 = 0$ and the fact that $G(y) \geq 0$ for all $y \in \mathbb{R}$.

Corollary 4.4. *Assume hypotheses (g0) and (g1).*
If $x : I \to \mathbb{R}$ is a solution of (1.1) then

$$G(x_1) + \int_{x_0}^{s} f(r) dr \geq 0 \quad \textit{for all } s \in x(I). \tag{4.2}$$

Equation (4.1) establishes a fundamental identity that solutions must fulfill. If $x : I \to \mathbb{R}$ is a solution and we put $t_1 = 0$ and $t_2 = t \in I$ in (4.1) we have

$$G(x'(t)) = G(x_1) + \int_{x_0}^{x(t)} f(r) dr \quad \text{for all } t \in I,$$

therefore, if $x' \geq 0$ on I we can have the solution expressed as a solution of the first-order problem

$$x'(t) = {G_{|[0,+\infty)}}^{-1} \left(G(x_1) + \int_{x_0}^{x(t)} f(r) dr \right), \quad x(0) = x_0,$$

and if $x' \leq 0$ on I then x would satisfy

$$x'(t) = {G_{|(-\infty,0]}}^{-1} \left(G(x_1) + \int_{x_0}^{x(t)} f(r) dr \right), \quad x(0) = x_0.$$

In general x' changes sign on I and thus there is no reason to expect x to be a solution of any one of the previous first-order problems. However we still can have another necessary condition of existence in terms of ${G_{|[0,+\infty)}}^{-1}$ and ${G_{|(-\infty,0]}}^{-1}$ if x is, respectively, *down-up* or *up-down*, in the sense of the following definition:

Definition 4.5. Let I be a nontrivial real interval and $\alpha \in C(I)$. We say that α is down-up on I if for each compact $K \subset x(I)$ there exist $t_1, t_2 \in I$ such that $t_1 < t_2$, $x(t_1) < x(t_2)$ and $K \subset x([t_1, t_2])$.

Analogously, we say that α is up-down on I if for each compact $K \subset x(I)$ there exist $t_1, t_2 \in I$ such that $t_1 < t_2$, $x(t_1) > x(t_2)$ and $K \subset x([t_1, t_2])$.

Remarks to Definition 4.5**.**

1. Nonconstant continuous mappings are down-up or up-down on their domains (or both, as it happens with the sine function on $\mathbb{R}$). To prove it let $\alpha \in C(I)$ be nonconstant. We can choose sequences $\{t_n\}$ and $\{s_n\}$ in I with respective limits $t_\infty, s_\infty \in \mathbb{R} \cup \{-\infty, +\infty\}$ and

$$\lim_{n\to\infty} \alpha(t_n) = \inf_{t\in I} \alpha(t) \in [-\infty, +\infty)$$

and

$$\lim_{n\to\infty} \alpha(s_n) = \sup_{t\in I} \alpha(t) \in (-\infty, +\infty].$$

If α attains a global minimum at some $t_0 \in I$ we simply take $t_n = t_0$ for all $n \in \mathbb{N}$, and we can proceed similarly if α attains a global maximum. Now it only remains to check whether $t_\infty < s_\infty$ or $t_\infty > s_\infty$ to deduce that α is, respectively, down-up or up-down (note that many possible choices of the sequences $\{t_n\}$ and $\{s_n\}$ are possible, but note also that a given function might be both down-up and up-down).

2. If a mapping $\alpha \in C^1(I)$ is down-up on I then $\alpha(I) = \alpha({\alpha'}^{-1}([0, +\infty)))$, i.e. every point in $\alpha(I)$ is reached at some $t \in I$ such that $\alpha'(t) \geq 0$. To prove it let $s \in \alpha(I)$ be fixed, consider the compact $K = \{s\}$, and let $t_1, t_2 \in I$ be such that $t_1 < t_2$, $x(t_1) < x(t_2)$ and $K \subset x([t_1, t_2])$. If $x(t_1) < s$ then $x(t^*) = s$ and $x'(t^*) \geq 0$ for

$$t^* = \sup\{t \in (t_1, t_2] \,:\, x(r) < s \text{ for all } r \in [t_1, t]\},$$

and if $x(t_2) > s$ then $x(t_*) = s$ and $x'(t_*) \geq 0$ for

$$t_* = \inf\{t \in [t_1, t_2) \,:\, x(r) > s \text{ for all } r \in [t, t_2]\}.$$

Analogously, if α is up-down on I then $\alpha(I) = \alpha({\alpha'}^{-1}((-\infty, 0]))$.

3. Nondecreasing (nonconstant) continuously differentiable mappings are down-up on their domains, and nonincreasing ones are up-down.

4. (Down-up/up-down solutions of (1.1)) Every solution of (1.1) with $x_1 > 0$ is increasing on a neighborhood of $t = 0$, therefore it is down-up on a neighborhood of $t = 0$. An analogous remark is valid for up-down solutions in connection with $x_1 < 0$. Therefore, from the local point of view, solutions of (1.1) can only be down-up when $x_1 > 0$ and can only be up-down when $x_1 < 0$.

However things are not so easy for noncontinuable solutions, as some problems (1.1) with $x_1 > 0$ may have up-down solutions that are not down-up and that cannot be extended to down-up solutions, or simply that cannot be extended at all. As an example, note that

$$x(t) = \begin{cases} \sin t, & \text{if } t \leq \pi, \\ \sqrt{2}\,\mathrm{tg}\left(-\frac{t-\pi}{\sqrt{2}}\right), & \text{if } \pi \leq t < \pi + \frac{\pi\sqrt{2}}{2}, \end{cases}$$

is a noncontinuable and up-down solution of (1.1) with $x_0 = 0$, $x_1 = 1$, $f(x) = -x$, and

$$g(y) = \begin{cases} -y, & \text{if } y \le -1, \\ 1, & \text{if } -1 < y. \end{cases}$$

To prove our next proposition we will need the following results on Lebesgue measure in $\mathbb{R}$, which will be denoted here and henceforth by μ. The second part of the lemma establishes that the set of the critical values of a everywhere differentiable function is null measurable.

Lemma 4.6. *Let $\alpha : [a, b] \to \mathbb{R}$ be a given mapping.*

1. [9, lemma 6.88] *Suppose that $M \subset (a, b)$ is measurable and that there exists $L \ge 0$ such that $D^+\alpha(x) \le L$ and $D_-\alpha(x) \ge -L$ for all $x \in M$. Then $\mu(\alpha(M)) \le L\mu(M)$.*
2. *If α is differentiable at every point of $[a, b]$, then $\mu(\alpha({\alpha'}^{-1}(\{0\}))) = 0$.*

Proof of part 2. Apply the first part to $M = {\alpha'}^{-1}(\{0\}) \setminus \{a, b\}$ with $L = 0$.

Next we establish our final necessary condition for the existence of nonconstant solutions. For simplicity we will use the following notation: $G_+^{-1} = G^{-1}_{|[0,+\infty)}$ and $G_-^{-1} = G^{-1}_{|(-\infty,0]}$.

Proposition 4.7. *Assume hypotheses (g0) and (g1) hold.*

If $x : I \to \mathbb{R}$ is a nonconstant solution of (1.1) then the mapping

$$F(s) := G(x_1) + \int_{x_0}^{s} f(r)dr \quad \text{for all } s \in x(I),$$

is well defined and

$$F(s) > 0 \quad \text{for almost all } s \in x(I). \tag{4.3}$$

Moreover, if x is down-up on I then

$$\frac{\max\{1, (g \circ G_+^{-1} \circ F)|f|\}}{G_+^{-1} \circ F} \in \mathrm{L}^1_{loc}(x(I)), \tag{4.4}$$

and for every $t_1, t_2 \in I$ with $t_1 \le t_2$ we have

$$\int_{x(t_1)}^{x(t_2)} \frac{dr}{G_+^{-1}(F(r))} \le t_2 - t_1; \tag{4.5}$$

and if, on the other hand, x is up-down on I then (4.4) is valid with G_+^{-1} replaced by G_-^{-1} and for every $t_1, t_2 \in I$ with $t_1 \le t_2$ we have

$$\int_{x(t_1)}^{x(t_2)} \frac{dr}{G_-^{-1}(F(r))} \le t_2 - t_1. \tag{4.6}$$

Proof. By (4.2) we know that $F(s) = G(x_1) + \int_{x_0}^{s} f(r)dr \geq 0$ for all $s \in x(I)$.

Let us suppose that x is down-up on I and define the measurable and nonnegative mapping $\phi : x(I) \to [0, +\infty]$ as

$$\phi(s) := \frac{1}{G_+^{-1}(F(s))}, \quad \text{if } F(s) > 0,$$
$$:= +\infty, \quad \text{if } F(s) = 0.$$

Note that ϕ is well-defined: since x is down-up, for each $s \in x(I)$ there exists $t \in I$ such that $x(t) = s$ and $x'(t) \geq 0$, therefore (4.1) with $t_1 = 0$ and $t_2 = t$ is equivalent to

$$G_{|[0,\infty)}(x'(t)) = G(x_1) + \int_{x_0}^{s} f(r)dr = F(s),$$

in particular, $G_+^{-1}(F(s))$ is defined.

Let $t_1, t_2 \in I$ be such that $t_1 < t_2$ and $x(t_1) < x(t_2)$, and denote $J = [t_1, t_2]$. We are going to prove that

$$\phi \text{ and the product } |(g \circ G_+^{-1} \circ F) f \phi| \text{ belong to } \mathrm{L}^1(x(J)). \tag{4.7}$$

Elementary arguments show that every point in $x(J)$ is attained at some point in J with nonnegative derivative, i.e.

$$x(J) = x(J \cap x'^{-1}([0, +\infty))) = x(J \cap x'^{-1}(\{0\})) \cup x(J \cap x'^{-1}(0, +\infty)),$$

and since, by virtue of the second part of lemma 4.6, we have

$$\mu(J \cap x(x'^{-1}(\{0\}))) = 0,$$

then (4.7) is equivalent to

$$\phi, \ |(g \circ G_+^{-1} \circ F) f \phi| \in L^1(x((t_1, t_2) \cap x'^{-1}(0, +\infty))), \tag{4.8}$$

as we are only removing null-measure subsets from $x(J)$.

The set $(t_1, t_2) \cap x'^{-1}(0, +\infty)$ is open, thus it can be expressed as a countable union of pairwise disjoint intervals, say $I_n = (a_n, b_n)$, $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be fixed. By definition of I_n and (4.1) with t_1 replaced by 0, for all $t \in I_n$ we have

$$0 < x'(t) = G_+^{-1}\left(G(x_1) + \int_{x_0}^{x(t)} f(r)dr \right) = \frac{1}{\phi(x(t))}, \tag{4.9}$$

and therefore $(\phi \circ x)x' = 1$ on I_n, so we can apply the first part of lemma 3.1 with f replaced by ϕ and I by $[a_n, b_n]$, to have that $\phi \in \mathrm{L}^1(x(a_n), x(b_n))$ and

$$\int_{x(a_n)}^{x(b_n)} \phi(r)dr = \int_{a_n}^{b_n} \phi(x(s))x'(s)ds = b_n - a_n.$$

Hence we have

$$\begin{aligned}\int_{x(x'^{-1}(0,+\infty)\cap(t_1,t_2))}\phi(r)dr &= \int_{\cup x(I_n)}\phi(r)dr\\ &\le \sum_{n=1}^{\infty}\int_{x(a_n)}^{x(b_n)}\phi(r)dr = \sum_{n=1}^{\infty}(b_n-a_n)\\ &\le t_2-t_1<+\infty,\end{aligned}$$

then we can apply the first part of lemma 3.1 with f replaced by ϕ and I by J, to have $\phi \in L^1(x(J))$ and

$$\int_{x(t_1)}^{x(t_2)}\phi(r)dr \le t_2-t_1. \tag{4.10}$$

Note that $\phi \in L^1(x(J))$ implies that (4.3) holds with $x(I)$ replaced by $x(J)$, and then (4.10) is equivalent to (4.5) (note that (4.5) is trivial if $x(t_1) \ge x(t_2)$).

To show that $h = |(g\circ G_+^{-1}\circ F)f\phi|$ satisfies (4.8), we first use (4.9) to ensure that for each $n \in \mathbb{N}$ the following relations hold on I_n:

$$(h\circ x)x' = g(G_+^{-1}(F(x)))|f(x)|\phi(x)x' = g(x')|f(x)| = |x''| \in L^1(I_n),$$

and then the first part of lemma 3.1 implies $h \in L^1(x(I_n))$ and

$$\begin{aligned}\int_{x(x'^{-1}(0,+\infty)\cap(t_1,t_2))}h(r) &= \int_{\cup x(I_n)}h(r)dr\\ &\le \sum_{n=1}^{\infty}\int_{x(a_n)}^{x(b_n)}h(r)dr\\ &= \sum_{n=1}^{\infty}\int_{a_n}^{b_n}|x''(s)|ds \le \int_{t_1}^{t_2}|x''(s)|ds<+\infty.\end{aligned}$$

Note that, since x is down-up on I, each compact $K \subset x(I)$ is contained in some $x(J)$ with J as above, and thus $\max\{1, g(G^{-1}(F(\cdot)))|f|\}\phi \in L^1_{loc}(x(I))$. This implies (4.3) and (4.4).

Finally, if x is up-down on I, then $v(t) := x(-t)$ for all $t \in -I := \{-t \,:\, t \in I\}$ is down-up on $-I$ and it is a solution of (1.1) with g replaced by $\tilde{g}(y) := g(-y)$ for all $y \in \mathbb{R}$ and x_1 replaced by $-x_1$. Therefore (4.3), (4.4) and (4.5) are valid with x_1 replaced by $-x_1$, $x(I)$ replaced by $v(-I)$, G replaced by

$$\tilde{G}(y) := \int_0^y \frac{r}{\tilde{g}(r)}dr \quad \text{for all } y \in \mathbb{R},$$

g replaced by $\tilde{g}$, and G_+^{-1} replaced by $\tilde{G}^{-1}_{|[0,+\infty)}$. Now it suffices to take into account that $x(I) = v(-I)$, $\tilde{G}(y) = G(-y)$ for all $y \in \mathbb{R}$ and $\tilde{G}^{-1}_{|[0,+\infty)} = -G_-^{-1}$ to conclude that (4.4) with G_+^{-1} replaced by G_+^{-1} and (4.6) hold. $\square$

5. Sufficient conditions for existence of solutions

Next we prove that the previous necessary conditions for existence of nonconstant solutions are also sufficient. We shall refer to the time maps

$$\tau_{\text{sgn}(x_1)} : y \in J \longmapsto \tau_{\text{sgn}(x_1)}(y) := \int_{x_0}^{y} \frac{dr}{{G_{\text{sgn}(x_1)}}^{-1}\left(G(x_1) + \int_{x_0}^{r} f(s)ds\right)}, \qquad (5.1)$$

where the notation $\text{sgn}(z)$ means $+$ when $z > 0$, $-$ when $z < 0$, and can be either $+$ or $-$ when $z = 0$.

Theorem 5.1. *Assume hypotheses (g0) and (g1).*

The problem (1.1) has the constant solution if and only if $f(x_0) = 0 = x_1$.

Moreover, the following statements are pairwise equivalent for a given nontrivial interval J such that $x_0 \in J$:

(i) $f \in L^1_{loc}(J)$, $G(x_1) + \int_{x_0}^{y} f(s)ds > 0$ *for almost all* $y \in J$, *and*

$$\frac{\max\{1, g\left({G_{sgn(x_1)}}^{-1}\left(G(x_1) + \int_{x_0}^{\cdot} f(s)ds\right)\right)|f|\}}{{G_{sgn(x_1)}}^{-1}\left(G(x_1) + \int_{x_0}^{\cdot} f(s)ds\right)} \in L^1_{loc}(J). \qquad (5.2)$$

(ii) *The problem (1.1) has a strictly monotone solution x implicitly given by*

$$\int_{x_0}^{x(t)} \frac{dr}{{G_{sgn(x_1)}}^{-1}\left(G(x_1) + \int_{x_0}^{r} f(s)ds\right)} = t \quad \text{for all } t \in \tau_{sgn(x_1)}(J), \qquad (5.3)$$

which increases if $sgn(x_1) = +$ and decreases if $sgn(x_1) = -$.

(iii) *The problem (1.1) has a nonconstant solution $x : I \to \mathbb{R}$ with $x(I) = J$ and $sgn(x_1)x$ is down-up on I.*

Proof. Obviously, $x(t) = x_0$ for all $t \in \mathbb{R}$ solves (1.1) if $f(x_0) = 0 = x_1$, and only in that case.

Let us show that (i) implies (ii). Assume first that $\text{sgn}(x_1) = -$ and consider the mapping τ_- defined in (5.1). The assumptions imply that τ_- is absolutely continuous over each compact subinterval of J and that ${\tau_-}'(y) < 0$ a.e. on J. Therefore lemma 3.4 ensures that it has a strictly decreasing inverse $x = {\tau_-}^{-1} : \tau(J) \to \mathbb{R}$ which is absolutely continuous over each compact subinterval of $\tau_-(J)$. Furthermore $\tau_-(x_0) = 0$ and thus $x(0) = x_0$.

Lemma 3.4 also ensures that

$$x'(t) = {G_-}^{-1}\left(G(x_1) + \int_{x_0}^{x(t)} f(s)ds\right) \quad \text{for a.a. } t \in \tau_-(J), \qquad (5.4)$$

and since the right-hand side is continuous then x' is continuous as well and equality holds everywhere in (5.4). In particular $x'(0) = x_1$.

Let us show now that x' is absolutely continuous on each compact subinterval of $\tau_-(J)$. Denote $F(s) = G(x_1) + \int_{x_0}^{s} f(r)dr$ for $s \in J$. Then

$$(((G_-{}^{-1})' \circ F)F')(s) = \frac{g(G_-{}^{-1}(F(s)))f(s)}{G_-{}^{-1}(F(s))} \quad \text{for a.a. } s \in J$$

and since by hypothesis it belongs to $\mathrm{L}^1_{loc}(J)$, lemma 3.2, (i), with $[a, b]$ replaced by any compact subinterval of J guarantees that $G_-{}^{-1} \circ F$ is absolutely continuous on such a compact interval. Now the absolute continuity of x' follows from the relation $x' = (G_-{}^{-1} \circ F) \circ x$, the monotonicity of x and [7, theorem 9.3].

Theorem 6.93 in [9] ensures the possibility of using the chain rule in (5.4) to have $x''(t) = f(x(t))g(x'(t))$ for a.a. $t \in \tau_-(J)$ (we note that $x' \neq 0$ a.e.). Hence x is an decreasing solution of (1.1).

In case $\mathrm{sgn}(x_1) = +$ it suffices to replace τ_- by τ_+.

Now (ii) trivially implies (iii), and (iii) implies (i) by virtue of propositions 4.2, 4.3 and 4.7. □

Remark 5.2. Existence of a solution of (1.1) with $x_1 \geq 0$ and range J implies the existence of a down-up solution with range J *when g is even.*

To prove it let $x : I \to \mathbb{R}$ be a solution of (1.1) with $x_1 \geq 0$ which is not down-up and we will show that there is a down-up solution with the same range. First since x must be up-down, there exists $t_0 \in I$ such that $x(t_0) = x_0$ and $x'(t_0) = -x_1$. On the other hand, since x is not down-up there exists $t_1 \in I$ such that $x'(t_1) = 0$. Let $v(t) = x(2t_1 - t)$ for $t \in \hat{I} := 2t_1 - I$. Since g is even, v is a solution of the differential equation, $x(I) = v(\hat{I})$, and

$$v(2t_1 - t_0) = x_0 \quad \text{and} \quad v'(2t_1 - t_0) = x_1,$$

so, up to translation in time, v and x are solutions of the same problem having the same range.

Similarly, the existence of solutions with $x_1 \leq 0$ and range J implies the existence of up-down solutions with the same range when g is even.

The previous remark, and the fact that $G_-{}^{-1} = -G_+{}^{-1}$ when g is even, allow us to prove the following corollary of theorem 5.1.

Corollary 5.3. *Assume hypotheses (g0) and (g1).*

If g is even then the following statements are pairwise equivalent for a given nontrivial interval J such that $x_0 \in J$:

(i) $f \in L^1_{loc}(J)$*,* $G(x_1) + \int_{x_0}^{y} f(s)ds > 0$ *for almost all* $y \in J$*, and*

$$\frac{\max\{1, g\left(G_+{}^{-1}\left(G(x_1) + \int_{x_0}^{\cdot} f(s)ds\right)\right)|f|\}}{G_+{}^{-1}\left(G(x_1) + \int_{x_0}^{\cdot} f(s)ds\right)} \in L^1_{loc}(J). \tag{5.5}$$

(ii) *The problem (1.1) has a strictly monotone solution x implicitly given by*

$$\int_{x_0}^{x(t)} \frac{dr}{{G_+}^{-1}\left(G(x_1)+\int_{x_0}^{r} f(s)ds\right)} = sgn(x_1)t \quad \text{for all } t \in sgn(x_1)\tau_+(J), \tag{5.6}$$

which increases if $sgn(x_1)=+$ and decreases if $sgn(x_1)=-$.

(iii) *The problem (1.1) has a nonconstant solution $x : I \to \mathbb{R}$ with $x(I)=J$.*

Remark 5.4. Taking $g \equiv 1$ the preceding corollary reduces to [6, theorem 3.1]

If we allow g to vanish the search of necessary conditions for the existence of solution of equation (1.1) seems to be much more difficult, as we already have pointed out. On the other hand a detailed revision of "$(i) \Longrightarrow (ii)$" in the proof of theorem 5.1 shows that we can obtain the same conclusion under weaker assumptions over g, as we present in the following theorem.

Theorem 5.5. *Suppose that the following assumptions hold:*

(h1) *$g(y)>0$ for a.a. $y \in \mathbb{R}$ and $\dfrac{\cdot}{g(\cdot)} \in \mathrm{L}^1_{loc}(\mathbb{R})$,*

(h2) *$f \in \mathrm{L}^1_{loc}(J)$, $G(x_1)+\int_{x_0}^{s} f(r)dr > 0$ for almost all $s \in J$, and*

$$\frac{\max\{1, g\left({G_{sgn(x_1)}}^{-1}\left(G(x_1)+\int_{x_0}^{\cdot} f(s)ds\right)\right)|f|\}}{{G_{sgn(x_1)}}^{-1}\left(G(x_1)+\int_{x_0}^{\cdot} f(s)ds\right)} \in \mathrm{L}^1_{loc}(J).$$

Then (1.1) has a strictly monotone solution x implicitly given by

$$\int_{x_0}^{x(t)} \frac{dr}{{G_{sgn(x_1)}}^{-1}\left(G(x_1)+\int_{x_0}^{r} f(s)ds\right)} = t \quad \text{for all } t \in \tau_{sgn(x_1)}(J).$$

6. Uniqueness, extremality and multiplicity

Whenever $x_1=0$, Theorem 5.1 (i) implies the existence of an increasing solution (considering $sgn(0)=+$) and a decreasing one (considering $sgn(0)=-$). If $f(x_0)=0$ we would also have the constant solution $x(t)\equiv x_0$. Thus, if $x_1=0$ we cannot expect in general to have local uniqueness for problem (1.1). However if $x_1 \neq 0$ the solution given by Theorem 5.1 is locally unique.

Theorem 6.1. *Assume hypotheses (g0) and (g1).*

If $x\colon I \to \mathbb{R}$ is a solution of (1.1) with $x_1 \neq 0$, then x is the unique solution of (1.1) on the connected component of $\{t \in I\colon x'(t) \neq 0\}$ that contains 0 (we shall denote it by I_+).

Proof. Let $x\colon I \to \mathbb{R}$ be a nonconstant solution of (1.1). Then $sgn(x_1)x$ is down-up in I_+ and Theorem (5.1) (i) holds for $J_+ = x(I_+)$. Formula (4.1) with $t_1=0$ and $t_2 = t \in I_+$ gives

$$0 < G\left(x'(t)\right) = G(x_1) + \int_{x_0}^{x(t)} f(r)\,dr \quad \text{for all } t \in I_+; \tag{6.1}$$

hence

$$\frac{x'(t)}{G_{sgn(x_1)}^{-1}\left(G(x_1)+\int_{x_0}^{x(t)} f(r)\,dr\right)}=1 \text{ for all } t\in I_+,$$

and Lemma 3.1 (1) ensures that integration between 0 and $t \in I_+$ gives

$$\tau_{sgn(x_1)}\left(x(t)\right)=t \text{ for all } t\in I_+. \tag{6.2}$$

In particular, $\tau_{sgn(x_1)}\left(x\left(I_+\right)\right)=I_+$.

Let $y\colon I_y \to \mathbb{R}$ be another solution of (1.1), such that $I_+\cap I_y$ is a nontrivial interval. If we prove that $y'(t)\neq 0$ for all $t\in I_+\cap I_y$, we obviously have $x(t)=y(t)$ in $I_+\cap I_y$, since $y(t)$ would satisfy $\tau_{sgn(x_1)}\left(y(t)\right)=t$.

Towards a contradiction, assume that there exists an interval $[a,b]\subset I_+\cap I_y$, such that $|y'(t)|>0$ for all $t\in(a,b)$, and $a=y'(b)=0$, or $y'(a)=b=0$. Let us consider the first case since the other one may be treated analogously.

By (4.1), we have

$$0=G(y'(b))=G(x_1)+\int_{x_0}^{y(b)} f(r)\,dr,$$

which implies, by (6.1), that $y(b)\notin x(I_+)$.

Using (4.1) again, we deduce that, for all $t\in[0,b)$,

$$\frac{y'(t)}{G_{sgn(x_1)}^{-1}\left(G(x_1)+\int_{x_0}^{y(t)} f(r)\,dr\right)}=1,$$

and integrating between 0 and b, we get

$$\tau_{sgn(x_1)}\left(y\left(b\right)\right)=b$$

and therefore, $\tau_{sgn(x_1)}(y(b))\in I_+=\tau_{sgn(x_1)}(x(I_+))$. By increasingness of $\tau_{sgn(x_1)}$, we get that $y(b)\in x(I_+)$, which is a contradiction. □

Combining the previous result with Theorem 5.1, we obtain an existence and uniqueness result.

Corollary 6.2. *Assume hypotheses (g0) and (g1) and moreover suppose that* $x_1\neq 0$ *and Theorem* 5.1*(i) holds.*

Then (5.3) *defines the unique solution of* (1.1) *on* $\tau_{sgn(x_1)}(J_+)$, *where* J_+ *is the connected component that contains* x_0 *of*

$$\left\{y\in J\colon G(x_1)+\int_{x_0}^{y} f(r)\,dr>0\right\}.$$

Proof. Theorem (5.1) guarantees that (5.3) defines a solution. On the other hand, the set $\tau_{sgn(x_1)}(J_+)$ is precisely the connected component which contains 0 of the set of timevalues where the derivative does not vanish, thus we get the conclusion applying Theorem 6.1. □

In case g is even Corollary 6.2 can be applied to deduce the following necessary and sufficient condition for global uniqueness of (5.3) on $\tau_{sgn(x_1)}(J)$ whenever $x_1 \neq 0$.

Theorem 6.3. *Assume hypotheses (g0) and (g1) and moreover suppose that g is an even function, $x_1 \neq 0$ and Theorem 5.1(i) holds.*

Then a necessary and sufficient condition for (5.3) to define the unique solution of (1.1) on the interval $\tau_{sgn(x_1)}(J)$ is

$$G(x_1) + \int_{x_0}^{y} f(r)\,dr > 0 \quad \text{for all } y \in J.$$

Proof. The sufficient condition is an immediate consequence of corollary 6.2.

To establish the necessity, assume by contradiction that there is some $y_0 \in J$ such that $G(x_1) + \int_{x_0}^{y_0} f(r)\,dr = 0$, and let t_0 be such that $t_0 = \tau_{sgn(x_1)}(y_0)$. If x denotes the solution given by (5.3), relation (4.1) gives

$$G(x'(t_0)) = G(x_1) + \int_{x_0}^{x(t_0)} f(r)\,dr = G(x_1) + \int_{x_0}^{y_0} f(r)\,dr = 0,$$

and therefore since g is even one can construct a different solution by reflecting the branch of the graph of x which passes through (t_0, x_0) by symmetry with respect to the line $t = t_0$. This is a contradiction with the uniqueness assumption. □

Now we study the case $x_0 = 0$.

Proposition 6.4. *Assume hypotheses (g0) and (g1) and suppose that $x_1 = 0 = f(x_0)$.*

Then (1.1) has only the constant solution if and only if Theorem 5.1 (i) does not hold.

A remarkable consequence of Proposition 6.4 is the following corollary.

Corollary 6.5. *Assume hypotheses (g0) and (g1) and suppose that $x_1 = 0 = f(x_0)$.*

Then (1.1) has only the constant solution provided that f is nonincreasing on a neighborhood of x_0.

Proof. Since $G(x_1) = 0$ we have $G(x_1) + \int_{x_0}^{y} f(r)\,dr \leq 0$ in a neighborhood of x_0, which means that (5.1) (i) does not hold. □

As we stated in the beginning of this section, if $x_1 = 0$ the problem (1.1) may have more than one solution. But in this situation, the two solutions given by Theorem 5.1 (considering $sgn(0) = +$ and $sgn(0) = -$) are extremal in some sense on each side of $t = 0$. Let us denote these solutions by

$$x_{\pm}(t) = \tau_{\pm}^{-1}(t), \quad \text{for all } t \in \tau_{\pm}(J).$$

Proposition 6.6. *Assume hypotheses (g0) and (g1) and suppose that $x_1 = 0$.*

The following assertions hold.

(i) *If* $x\colon I\to\mathbb{R}$ *is a down-up solution of* (1.1) *then:*
$x(t)\le x_+(t)$ *for all* $t\in I\cap\tau_+(J)\cap[0,+\infty)$*, and*
$x(t)\ge x_+(t)$ *for all* $t\in I\cap\tau_+(J)\cap(-\infty,0]$.

(ii) *If* $x\colon I\to\mathbb{R}$ *is an up-down solution of* (1.1) *then:*
$x(t)\ge x_-(t)$ *for all* $t\in I\cap\tau_-(J)\cap[0,+\infty)$*, and*
$x(t)\le x_-(t)$ *for all* $t\in I\cap\tau_-(J)\cap(-\infty,0]$.

(iii) *If* g *is even and* $x\colon I\to\mathbb{R}$ *is any solution of* (1.1) *then the conclusions of (i) and (ii) follow.*

Proof. We shall prove part (i). If x is a down-up solution by (4.5) we have for all $t\in I\cap\tau_+(J)\cap[0,+\infty)$ that

$$\int_{x_0}^{x(t)}\frac{dr}{G_+^{-1}(F(r))}=\tau_+(x(t))\le t,$$

or equivalently, $x(t)\le x_+(t)$. On the other hand, for all $t\in I\cap\tau_+(J)\cap(-\infty,0]$, we have $\tau_+(x(t))\ge t$, which implies $x(t)\ge x_+(t)$.

Part (ii) admits an analogous treatment and finally for part (iii) it suffices to note that for an even g we have that $G_+^{-1}=-G_-^{-1}$. □

Now let us present necessary and sufficient conditions which guarantee that all the nontrivial solutions of (1.1) with $x_1=0$ are exactly the ones given in (5.3) on each side of $t=0$.

Theorem 6.7. *Assume hypotheses (g0), (g1) and suppose that* g *is even, Theorem* 5.1*(i) holds,* $x_1=0$ *and* $x_0\in Int(J)$.

Then the following assertions are pairwise equivalent:

(i) *For all* $y\in Int(J)\setminus\{x_0\}$*, we have*

$$\int_{x_0}^{y}f(r)\,dr>0. \tag{6.3}$$

(ii) *If* $x\colon[0,T]\to\mathbb{R}$ *is a solution of* (1.1) *which is is not constant on* $[0,\delta]$*, for all* $\delta\in(0,T]$*, then either* $x=x_+$ *on* $[0,T]\cap\tau_+(J)$*, or* $x=x_-$ *on* $[0,T]\cap\tau_-(J)$.

(iii) *For all* $t\in\tau_\pm(Int(J))\cap(0,+\infty)$*, we have* $x_\pm'(t)\ne 0$.

Proof. *(i)* ⇒ *(ii)* Let $x\colon[0,T]\to\mathbb{R}$ be a solution of (1.1) which is is not constant on $[0,\delta]$, for all $\delta\in(0,T]$, and let $\hat{T}\in(0,T]$ be such that $x(t)\in Int(J)$ for all $t\in[0,\hat{T})$.

We claim that $x'(t)\ne 0$, for all $t\in(0,\hat{T})$. Towards a contradiction, assume that there exists some $t_0\in(0,\hat{T})$ such that $x'(t_0)=0$. Hence (4.1) yields

$$0=G(x'(t_0))=\int_{x_0}^{x(t_0)}f(r)\,dr,$$

and, using (6.3) and the fact that $x(t_0)\in Int(J)$, we have that $x(t_0)=x_0$. Now, since x is not constant on $[0,t_0]$, there exists $t_1\in(0,t_0)$ such that $x(t_1)\ne x_0$ and

$x'(t_1) = 0$. Then

$$0 = G(x'(t_1)) = \int_{x_0}^{x(t_1)} f(r)\,dr,$$

and since $x(t_1) \in Int(J) \setminus \{x_0\}$, we have a contradiction with (6.3).

There are now two possibilities: either $x' > 0$ on $(0, \hat{T})$, or $x' < 0$ on $(0, \hat{T})$.

If $x' > 0$ on $(0, \hat{T})$, then (4.1) implies that

$$\frac{x'(t)}{G_+^{-1}\left(\int_{x_0}^{x(t)} f(s)\,ds\right)} = 1 \text{ on } (0, \hat{T}),$$

so we can integrate between 0 and $t \in (0, \hat{T}]$, to conclude that

$$\tau_+(x(t)) = t \text{ for all } t \in [0, \hat{T}].$$

Thus we have proven that $x(t) = x_+(t)$ whenever $t \geq 0$ and $x(t) \in Int(J)$. Since $x_+(t) = \tau_+^{-1}(t) \in Int(J)$ for all $t \in \tau_+(Int(J))$, we conclude that $x = x_+$ on $\tau_+(Int(J)) \cap [0, T]$, and, by continuity of x and x_+, that $x = x_+$ on $\tau_+(J) \cap [0, T]$.

On the other and, if $x' < 0$ on $(0, \hat{T})$, then we obtain

$$\tau_-(x(t)) = t \text{ for all } t \in [0, \hat{T}],$$

and we can conclude in a similar way that $x = x_-$ on $\tau_-(J) \cap [0, T]$.

(ii) $\Rightarrow$ *(iii)* Assume, reasoning by contradiction, that there exists some $t_0 \in \tau_+(Int(J)) \cap (0, +\infty)$ such that $x'_+(t_0) = 0$. Extending the restriction of x_+ to $[0, t_0]$ by symmetry with respect to $t = t_0$, we obtain a new nonconstant solution, which is different from x_+ because it is not monotone on $\tau_+(J)$, and different from x_- because it is increasing on $[0, t_0]$, which is a contradiction with (ii). We get a similar contradiction if we assume that $x'_-(t_0) = 0$ for some $t_0 \in \tau_-(Int(J)) \cap (0, +\infty)$

(iii) $\Rightarrow$ *(i)* For each $t \in \tau_+(Int(J)) \cap (0, +\infty)$, we have

$$0 < G(x'_+(t)) = \int_{x_0}^{x_+(t)} f(r)\,dr,$$

and therefore $\int_{x_0}^{y} f(r)\,dr > 0$ for all $y \in x_+(\tau_+(Int(J)) \cap (0, +\infty)) = Int(J) \cap (x_0, +\infty)$. An analogous argument with x_- yields $\int_{x_0}^{y} f(r)\,dr > 0$ for all $y \in Int(J) \cap (-\infty, x_0)$. □

Remark 6.8. An analogous result holds for solutions $x\colon [-T, 0] \to \mathbb{R}$, with $T > 0$.

We have the following similar result for the case in which x_0 the minimum of J, and J cannot be extended. The similar case with x_0 being the maximum of J can be treated analogously.

Theorem 6.9. *Assume hypotheses (g0), (g1) and suppose that g is even.*

Let $x_1 = 0$ and suppose that there exists $\epsilon > 0$ such that Theorem 5.1 *(i) holds for $J = [x_0, x_0 + \epsilon)$ and not for $(x_0 - \rho, x_0 + \epsilon)$ for any $\rho > 0$. Then the following assertions are pairwise equivalent:*

(i) *For all $y \in (x_0, x_0+\epsilon)$, we have*
$$\int_{x_0}^{y} f(r)\,dr > 0. \tag{6.4}$$
(ii) *If $x\colon [0,T] \to \mathbb{R}$ is a solution of (1.1) which is not constant on $[0,\delta]$, for all $\delta \in (0,T]$, then $x = x_+$ on $[0,T] \cap \tau(J)$.*
(iii) *For all $t \in \tau_+(x_0, x_0+\epsilon) \cap (0,+\infty)$, we have $x_+' > 0$.*

Proof. To prove that (i) implies (ii), let $x\colon [0,T] \to \mathbb{R}$ be a solution of (1.1) which is not constant on $[0,\delta]$, for all $\delta \in (0,T]$. Note that $x(t) \ge x_0$ for all $t \in [0,T]$, since otherwise Theorem 5.1 implies that (5.1 (i)) holds for $(\min_{t\in[0,T]} x(t), x_0]$, which is a contradiction. Thus there exists $\hat{T} \in (0,T]$ such that $x(t) \in J$ for all $t \in [0,\hat{T}]$. Now it suffices to follow the proof of Theorem 6.7 to show that $x' \neq 0$ on $(0,\hat{T})$. Since $x \ge x_0$ on $[0,T]$, we deduce that $x' > 0$ on $(0,\hat{T})$, and obtain $x = x_+$ on $[0,T] \cap \tau(J)$. The remaining parts of the proof follow analogously to the ones in Theorem 6.7. □

We show now that, under stronger assumptions than those in Theorem 5.1 (i), we can obtain solutions in the case $x_1 = 0$ which are non constant or monotone on $[0,\delta]$ and on $[-\delta,0]$, for each sufficiently small $\delta > 0$.

Proposition 6.10. *Assume hypotheses (g0), (g1) and suppose that g is even.*

Let $x_1 = 0$ and suppose that f and g satisfy Theorem 5.1 (i) with $J = [x_0, x_0+\epsilon]$ for some $\epsilon > 0$.

Moreover assume that $f \in L^\infty(J)$ and that $(y_n)_{n\in\mathbb{N}}$ is a decreasing sequence in J which converges to x_0 such that $\int_{x_0}^{y_n} f(r)\,dr = 0$ for all $n \in \mathbb{N}$, and
$$\sum_{n=1}^{\infty} \int_{x_0}^{y_n} \frac{dr}{G_+^{-1}\left(\int_{x_0}^{r} f(s)\,ds\right)} < +\infty.$$
Then problem (1.1) with $x_1 = 0$ has a solution which is not constant or monotone on $[0,\delta]$ and on $[-\delta,0]$, for all $\delta > 0$ small enough.

Proof. For each $n \in \mathbb{N}$ let x_n be the solution implicitly given by
$$\int_{x_0}^{x_n(t)} \frac{dr}{G_+^{-1}\left(\int_{x_0}^{r} f(s)\,ds\right)} = t, \text{ for } t \in \left[0, t_n := \int_{x_0}^{y_n} \frac{dr}{G_+^{-1}\left(\int_{x_0}^{r} f(s)\,ds\right)}\right]. \tag{6.5}$$

Since $x_n'(t_n) = 0$, x_n can be extended to the interval $[0, 2t_n]$ by symmetry with respect to the line $t = t_n$, and we find a solution $\tilde{x}_n$ defined on $[0,2t_n]$ which satisfies
$$\tilde{x}_n(0) = x_0 = \tilde{x}_n(2t_n), \text{ and } \tilde{x}_n'(0) = 0 = \tilde{x}_n'(2t_n).$$

Since our problem is autonomous and g is even, the translations $\tilde{x}_n(\cdot - \tau)$, for any $\tau \in \mathbb{R}$, are again solutions on $[\tau, \tau + 2t_n]$. Thus a new solution x :

$[0, T := 2\sum_{n=1}^{\infty} t_n] \to \mathbb{R}$ can be defined as follows:

$$x(t) = \begin{cases} \tilde{x}_1(t), & t \in [0, 2t_1], \\ \tilde{x}_n\left(t - 2\sum_{k=1}^{n-1} t_k\right), & t \in \left(2\sum_{k=1}^{n-1} t_k, 2\sum_{k=1}^{n} t_k\right], \\ x_0, & t = 2\sum_{k=1}^{\infty} t_k. \end{cases}$$

Let us show that x is in fact a solution. The continuity of x is obvious for $t \in [0, T)$, and to prove the continuity at $t = T$, we just need to take into account the fact that

$$x_0 \leq x(t) \leq y_n, \text{ for all } t \in \left[2\sum_{k=1}^{n-1} t_k, 2\sum_{k=1}^{n} t_k\right], \text{ for each } n \in \mathbb{N}.$$

On each interval $\left[2\sum_{k=1}^{n-1} t_k, 2\sum_{k=1}^{n} t_k\right]$ it it obvious that x' is absolutely continuous, and that the differential equation is satisfied. Let us show that $x' \in AC([0, T])$.

By virtue of (4.1), we have

$$G(x'(t)) = \int_{x_0}^{x(t)} f(r)\, dr, \text{ for all } t \in \left[2\sum_{k=1}^{n-1} t_k, 2\sum_{k=1}^{n} t_k\right],$$

and since, by construction, x' is continuous on $[0, T)$, we have

$$G(x'(t)) = \int_{x_0}^{x(t)} f(r)\, dr, \text{ for all } t \in [0, T). \tag{6.6}$$

Thus $\lim_{t\to T^-} x'(t) = 0$, and since x is continuous at $t = T$, we have $x'(T) = 0$.

On the other hand, the mapping $\int_{x_0}^{\cdot} f(r)\, dr$ is lipschitzian on $[x_0, y_1]$ because $f \in L^\infty(x_0, y_1)$, and therefore, [7, Theorem 9.3] together with (6.6) yields $G(x'(t)) \in AC[0, T]$. Finally, an application of Lemma 3.4 gives $x' \in AC[0, T]$.

To have a solution y in the conditions of the statement, we only need to extend x by symmetry with respect to the line $t = T$, say $\tilde{x} : [0, 2T] \to \mathbb{R}$, and consider the translation $y(t) = \tilde{x}(t + T)$, for $t \in [-T, T]$. □

7. Continuation and periodic solutions

Throughout this section we shall assume that Theorem 5.1 holds, and therefore, the problem (1.1) has strictly monotone solutions, given by (5.3). We are going to study the continuation of increasing solutions for the case $x_1 \geq 0$ and on the right side of $t = 0$. The remaining situations admit an analogous study.

First we need to have solutions defined on the right side of $t = 0$, so we shall assume that $J \cap [x_0, +\infty)$ is a nontrivial interval. In this case Theorem 5.1 guarantees that (1.1) with $x_1 \geq 0$ has a solution given by

$$x_+(t) = \tau_+^{-1}(t) \text{ for all } t \in \tau(J), \tag{7.1}$$

which is increasing on $\tau_+(J)$ and $\tau_+(J) \cap [x_0, +\infty)$ is a nontrivial interval.

We have several possibilities:

(i) Solution (7.1) is defined and unbounded on $[0,+\infty)$, provided that $[x_0,+\infty)\subset J$ and $\tau_+([x_0,+\infty))=[0,+\infty)$, that is

$$\int_{x_0}^{+\infty}\frac{dr}{G_+^{-1}\left(G(x_1)+\int_{x_0}^{r}f(s)\,ds\right)}=+\infty.$$

(ii) Solution (7.1) is defined and bounded on $[0,+\infty)$, provided that $J\cap[x_0,+\infty)=[x_0,\hat{x}_0)$ for some $\hat{x}_0>x_0$ such that

$$\lim_{x\to\hat{x}_0^-}\tau_+(x)=+\infty,$$

that is

$$\int_{x_0}^{\hat{x}_0}\frac{dr}{G_+^{-1}\left(G(x_1)+\int_{x_0}^{r}f(s)\,ds\right)}=+\infty.$$

(iii) Solution (7.1) blows up at finite time, provided that $[x_0,+\infty)\subset J$ and $\tau_+([x_0,+\infty))$ is bounded. If we define $T:=\sup\tau_+(J)\cap[0,+\infty)$ then we have that $\lim_{t\to T^-}x_+(t)=+\infty$. Note that $\tau_+([x_0,+\infty))$ is bounded if and only if

$$\int_{x_0}^{+\infty}\frac{dr}{G_+^{-1}\left(G(x_1)+\int_{x_0}^{r}f(s)\,ds\right)}<+\infty.$$

(iv) Solution (7.1) stops at finite time and at finite position, provided that $J\cap[x_0,+\infty)$ and $\tau_+(J)\cap[0,+\infty)$ are bounded. Defining $S:=\sup J\cap[x_0,+\infty)$ and $T:=\sup\tau_+(J)\cap[0,+\infty)$, we have

$$\lim_{t\to T^-}x_+(t)=S,$$

and the solution cannot be extended further by means of (7.1) on the right side of T because τ_+ is not defined on the right side of S.

Proposition 7.1. *Assume hypotheses (g0) and (g1) and suppose that Theorem* 5.1*(i) holds and that $J\cap[x_0,+\infty)$ is a nontrivial interval.*

*Then solution (*7.1*) can be continued over $[0,+\infty)$ as a constant on $[T,+\infty)$ for some $T\in\tau_+(J)\cap[0,+\infty)$ if and only if the set*

$$C=\left\{y_0\in J\cap[x_0,+\infty)\colon f(y_0)=0\ \ \textit{and}\ \ G(x_1)+\int_{x_0}^{y_0}f(r)\,dr=0\right\}\neq\emptyset$$

Proof. We only need to define $x(t)$ by (7.1) for $t\in[0,\tau_+(y_0))$, and $x(t)=y_0$ for $t\geq\tau_+(y_0)$, where y_0 is an element of C. Differentiability at $t=\tau_+(y_0)$ follows from $G(x_1)+\int_{x_0}^{y_0}f(r)\,dr=0$.

The converse is a trivial consequence of (4.1). □

Proposition 7.2. *Assume hypotheses (g0) and (g1) and suppose that g is an even function, that Theorem* 5.1*(i) holds and that $J\cap[x_0,+\infty)$ is a nontrivial interval.*

Suppose, moreover, that there exist $y_0\in J\cap[x_0,+\infty)$ and $y_1\in J\cap(-\infty,y_0)$, such that $G(x_1)+\int_{x_0}^{y_i}f(r)\,dr=0$, for $i=0,1$.

*Then problem (*1.1*) with* $x_1 \geq 0$ *has a solution on* $[x_0, +\infty)$*, which is increasing on* $[0, \tau_+(y_0))$ *and periodic on* $[\tau_+(y_0), +\infty)$*, of period* $2(\tau_+(y_0) - \tau_+(y_1))$*.*

Proof. First note that assumptions imply that the solution x_+ given by (7.1) satisfies $x'_+(\tau_+(y_1)) = x'_+(\tau_+(y_0)) = 0$.

Consider the restriction of x_+ to the interval $\tau_+(J) \cap (-\infty, \tau_+(y_0)]$ and extend it by symmetry with respect to the line $t = \tau_+(y_0)$. Now let $\tilde{x}$ be the restriction of the resulting solution to $[0, 2\tau_+(y_0) - \tau_+(y_1)]$ and note that $\tilde{x}'(2\tau_+(y_0) - \tau_+(y_1)) = x'_+(y_1) = 0$. Now take the restriction of $\tilde{x}$ to the interval $[\tau_+(y_0), 2\tau_+(y_0) - \tau_+(y_1)]$, extend it by symmetry with respect to the line $t = 2\tau_+(y_0) - \tau_+(y_1)$ over the interval $[2\tau_+(y_0) - \tau_+(y_1), 3\tau_+(y_0) - 2\tau_+(y_1)]$, and define $\tilde{x}$ like this on this interval. Finally, iterating this construction, we get the result. □

Theorem 7.3. *Assume hypotheses (g0) and (g1) and suppose that g is an even function.*

Let $f\colon Dom(f) \subset \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ *and* $m, M \in \mathbb{R}$ *be such that* $m < M$*.*

Then the following assertions are pairwise equivalent:

(i) $x'' = f(x)g(x')$ *has a solution for wich* m *and* M *are critical values.*

(ii) $f \in L^1(m, M)$*,* $\int_m^y f(r)\,dr > 0$ *for almost all* $y \in [m, M]$*,* $\int_m^M f(r)\,dr = 0$*, and*
$$\frac{\max\left\{1, g\left(G_+^{-1}\left(\int_m^{\cdot} f(s)\,ds\right)\right)|f|\right\}}{G_+^{-1}\left(\int_m^{\cdot} f(s)\,ds\right)} \in L^1(m, M).$$

(iii) $x'' = f(x)g(x')$ *has a periodic solution* $x(t)$*, with* $x(0) = m$*,* $x(T) = M$ *and period* $2T$*, where*
$$T = \int_m^M \frac{dr}{G_+^{-1}\left(\int_m^r f(s)\,ds\right)}.$$

Moreover x *increases on* $[0, T]$ *and decreases on* $[T, 2T]$*.*

Proof. $(i) \Rightarrow (ii)$ We can suppose without loss of generality that x is a down-up solution on $[0, T]$, $x(0) = m$, $x(T) = M$ for some $T \neq 0$ and $x'(0) = 0 = x'(T)$. Now it suffices to apply Propositions 4.2-4.7 to this solution, and note that $[m, M] \subset x([0, T])$. Finally we deduce from (4.1) with $t_1 = 0$ and $t_2 = T$ that $\int_m^M f(r)\,dr = 0$.

$(ii) \Rightarrow (iii)$ If we apply Proposition 7.2 to (1.1) with $J = [m, M]$, $x_0 = y_1 = m$, $x_1 = 0$ and $y_0 = M$, and we note that it can be extended periodically backwards by symmetry with respect to $t = 0$.

The implication $(iii) \Rightarrow (i)$ is clearly true. □

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