UNIQUENESS AND EXISTENCE RESULTS
FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

We establish some uniqueness and existence results for first-order ordinary differential equations with constant-signed discontinuous nonlinear parts. Several examples are given to illustrate the applicability of our work.

1. Introduction

In this paper we introduce conditions which ensure either at most one or exactly one Carathéodory type solution for the initial value problem (IVP)

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0, \tag{1}$$

where $f \colon I \times J \to \mathbb{R}$, $I = [t_0, t_0 + T]$, $J = [x_0, x_0 + R]$, $T > 0$ and $R > 0$.

In their fundamental paper [4] Hassan and Rzymowski were able to prove existence results for (1) with such assumptions which don't imply sup-measurability of f but avoid "downward jumps" for $f(t, \cdot)$. These assumptions have been generalized in [1,7] so that they allow other types of discontinuities.

In [3, 8] a new technique to study the case when f is nonnegative-valued is presented. It shows that if $f(t, x)$ is positive for a.a. t and all x, the inverse of a solution of (1) solves a reciprocal problem, defined later on. Applying this "inverse method" many new existence theorems have been proved in [3] to problem (1). Roughly speaking, the hypotheses of these theorems differ from standard ones because measurability of $f(\cdot, x)$ is replaced by measurability of $f(t, \cdot)$, and hypotheses imposed on $f(t, \cdot)$ are assumed for $f(\cdot, x)$.

Continuing the work started in [2], alternative types of new uniqueness results will now be proved for the IVP (1). These results are then combined with some existence

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theorems proved in [1] by "direct method", and in [3] by inverse method, to derive new existence and uniqueness theorems for differential equations with discontinuous and nonnegative right-hand sides. These theorems imply, for instance, that *the IVP (1) has exactly one solution if the values of f are positive and bounded above by R/T, and if f is monotone nonincreasing or nondecreasing with respect to both of its arguments.* Notice that no continuity hypotheses are imposed on the function f.

Examples are presented to illustrate the obtained results.

2. UNIQUENESS RESULTS

By a *lower solution* of (1) we mean an absolutely continuous function $x \colon I \to J$ such that $x'(t) \leq f(t, x(t))$ for a.e. $t \in I$, and $x(t_0) \leq x_0$. An *upper solution* is defined similarly, by reversing above inequalities. If equalities hold, we say that x is a *solution* of (1).

The following auxiliary result is a consequence of [3, Theorem 2.1].

Lemma 1. *Suppose that $f \colon I \times J \to \mathbb{R}$ satisfies the following hypothesis.*

(f0) $f(t, x) > 0$ *for a.e. $t \in I$ and all $x \in \mathbb{R}$.*

Define a function $\tilde{f} \colon I \times J \to \mathbb{R}$ by

$$\tilde{f}(t, x) = \begin{cases} \frac{1}{f(t,x)}, & \text{if } f(t,x) \neq 0, \\ 0, & \text{if } f(t,x) = 0. \end{cases} \tag{2}$$

If the reciprocal problem

$$t'(x) = \tilde{f}(t(x), x), \qquad t(x_0) = t_0, \tag{3}$$

has at most one solution on $[x_0, \alpha]$ for each $\alpha \in (x_0, x_0 + R]$, then problem (1) has at most one solution on $[t_0, \beta]$ for each $\beta \in (t_0, t_0 + T]$.

Our main uniqueness theorem is a complementary version to [6, Lemma 1.5.4].

Theorem 1. *The initial value problem (1) has at most one solution if $f \colon I \times J \to \mathbb{R}$ satisfies the hypothesis (f0) and the following hypotheses.*

(f1) $0 < M(x) \leq f(t, x)$ *for all $t \in I$ and for a.e. $x \in J$, where $\frac{1}{M} \in L^1(J)$.*

(f2) $f(s, x) - f(t, x) \leq g(t - s, x)$ *for all $s, t \in I$, $s \leq t$, and for a.e. $x \in J$, where*

(g0) $g \colon \mathbb{R}_+ \times J \to \mathbb{R}_+$, *and $\tau(x) \equiv 0$ is the only lower solution of the IVP*

$$\tau'(x) = \min\left\{ \frac{g(\tau(x), x)}{M^2(x)}, \frac{1}{M(x)} \right\} \quad \text{for a.e. } x \in J, \qquad \tau(x_0) = 0. \tag{4}$$

Proof. According to Lemma 1 it suffices to show that (3) has a unique solution on $[x_0, \alpha]$ for each $\alpha \in (x_0, x_0 + R]$.

The definition (2) of $\tilde{f}$ and the hypotheses (f1) and (f2) imply that

$$\tilde{f}(t, x) - \tilde{f}(s, x) = \frac{1}{f(t,x)} - \frac{1}{f(s,x)} = \frac{f(s,x) - f(t,x)}{f(t,x)f(s,x)} \leq \min\left\{ \frac{g(t - s, x)}{M^2(x)}, \frac{1}{M(x)} \right\} \tag{5}$$

for all $s, t \in I$, $t \leq s$, and for a.e. $x \in J$. Assume next that s, t are two solutions of (3) on $[x_0, \alpha]$, where $x_0 < \alpha \leq x_0 + R$. Because the solution set of (3) is directed, we may assume that $s(x) \leq t(x)$ for each $x \in [x_0, \alpha]$. Denoting

$$u(x) = \begin{cases} t(x) - s(x), & x \in [x_0, \alpha), \\ t(\alpha) - s(\alpha), & x \in [\alpha, x_0 + R], \end{cases}$$

it follows from (3) and (5) that u is is a lower solution of (4). This result and condition (g0) imply that $u(x) \equiv 0$. Thus $s(x) = t(x)$ for each $x \in [x_0, \alpha]$, which concludes the proof. $\qquad\qquad\square$

In the next result we present some special cases to Theorem 1.

Proposition 1. *The IVP (1) has at most one solution if the hypotheses of Theorem 1 hold with (g0) replaced by one of the following conditions:*

(g1) $g \colon \mathbb{R}_+ \times J \to \mathbb{R}_+$, $g(\tau, \cdot)$ *is measurable for all $\tau \geq 0$, for each $(\tau, x) \in [x_0, x_0 + R) \times \mathbb{R}_+$ there exist $\delta > 0$ and $\epsilon > 0$ such that*

$$\limsup_{s \to t-} g(s, y) \leq g(t, y) \text{ for a.e. } y \in [x, x + \delta] \text{ and all } t \in (\tau, \tau + \epsilon], \text{ and}$$

$$g(t, y) \leq \liminf_{s \to t+} g(s, y) \text{ for a.e. } y \in [x, x + \delta] \text{ and all } t \in [\tau, \tau + \epsilon),$$

and $\tau(x) \equiv 0$ is the only solution of the IVP (4).

(g2) $g \colon \mathbb{R}_+ \times J \to \mathbb{R}_+$, $g(\tau, \cdot)$ *is measurable for all $\tau \geq 0$, $g(\cdot, x)$ is monotone nondecreasing or continuous for a.e. $x \in J$, and $\tau(x) \equiv 0$ is the only solution of the IVP (4).*

Proof. To prove that (g1) implies (g0), assume that (g1) holds. Since $\frac{1}{M} \in L^1(J)$ by (f1), we can define

$$w(x) = \int_{x_0}^{x} \frac{ds}{M(s)}, \qquad x \in J. \tag{6}$$

Then $0 \leq w$ in $AC(J)$, ordered pointwise, 0 is a lower solution of (4) and w is its upper solution. Moreover, it is easy to see that $0 \leq u \leq w$ for each solution of (4). It then follows from [1,Theorem 2.1.4] that the IVP (4) has the greatest solution u^*, which is also the greatest of all the lower solutions of (4). Condition (g1) implies that $u^*(t) \equiv 0$, whence (g0) holds.

Condition (g2) is a special case of (g1), so that (g0) holds also in this case. $\qquad\square$

New versions of one-sided Osgood and Lipschitz-type uniqueness conditions are included in the next Corollary. The first-one generalizes the alternative version of two-sided Osgood-criterion presented in [2, Remark 3.1].

Corollary 1. *The IVP (1) has at most one solution if the hypotheses (f0), (f1) of Theorem 1 hold, and if condition (g0) in the hypothesis (f2) is replaced by one of the following conditions, where M is the function in the hypothesis (f1).*

(g3) $g(\tau, x) = \varphi(\tau) p(x)$, $x \in J$, $\tau \geq 0$, *where $p/M^2 \in L^1(J, \mathbb{R}_+)$, $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ is monotone nondecreasing, $\varphi(\tau) > 0$ when $\tau > 0$, and $\int_{0+}^{1} \frac{d\tau}{\varphi(\tau)} = \infty$;*

(g4) $g(\tau, x) = \max\{\tau \ln \frac{1}{\tau} \cdots \ln_n \frac{1}{\tau}, \tau \exp_n(1)\} \begin{cases} p(x) & \tau > 0, \ x \in J, \\ 0, & \tau = 0, \ r \in J, \end{cases}$ where $\ln_n$ and
$\exp_n$ denote the n-fold iterated logarithm and exponential function, respectively, and $p/M^2 \in L^1(J, \mathbb{R}_+)$;

(g5) $g(\tau, x) = \tau p(x)$, $\tau \geq 0$, $x \in J$, where $p/M^2 \in L^1(J, \mathbb{R}_+)$.

Proof. If (g3) holds, then the function $g(\tau, x) = \varphi(\tau)p(x)$, $\tau \in \mathbb{R}_+$, $x \in J$, has the properties given for g in (g2). Thus (g2), and hence also (g0), holds.

The functions

$$\varphi(\tau) = \tau, \ \tau \geq 0 \ \text{ and } \ \varphi(\tau) = \begin{cases} \max\{\tau \ln \frac{1}{\tau} \cdots \ln_n \frac{1}{\tau}, \tau \exp_n(1)\}, & \tau > 0, \\ 0, & \tau = 0, \end{cases} \ n = 1, 2, \ldots$$

satisfy the hypotheses given for φ in (g3). Thus (g4) and (g5) are special cases of (g3). $\square$

Remarks 1. Measurability of $\frac{1}{M}$ is not needed in the proof of Theorem 1.

It follows from [5, Theorem 2.1] and from (5) that the result of Theorem 1 holds also when (g0) is replaced by

(g6) $g: \mathbb{R}_+ \times J \to \mathbb{R}_+$, $g(u(\cdot), \cdot)/M^2 \in L^1(J)$ whenever $u \in AC(J, \mathbb{R}_+)$, and $\tau(x) \equiv 0$ is the only absolutely continuous solution of the integral inequality

$$\tau(x) \leq \int_{x_0}^x \min\left\{\frac{g(\tau(s), x)}{M^2(s)}, \frac{1}{M(s)}\right\} ds. \tag{7}$$

The function M^2 cannot be dropped out from (4) and (7) even if it is a constant. For instance, given $0 < M < 1$, define

$$g(\tau, x) = \begin{cases} 2M^2 x, & \tau \geq x^2, \ 0 \leq x \leq 1, \\ \frac{2M^2 \tau}{x}, & 0 < \tau < x^2, \ 0 < x \leq 1. \end{cases}$$

Then $\tau(x) = x^2$ is a nonzero solution of both (4) and (7) on $[0, 1]$, when $x_0 = t_0 = 0$, whence the hypotheses (g0) and (g6) are not valid. On the other hand, if M^2 is replaced by 1 in (4) and (7), their only solution is zero function.

3. EXISTENCE AND UNIQUENESS RESULTS

In this section we combine the uniqueness results derived in Section 3 to existence results proved in [1,3] to obtain new existence and uniqueness results for the IVP (1).

Theorem 2. *The initial value problem (1) has exactly one absolutely continuous solution if f satisfies the hypotheses (f0), (f1) and (f2), combined with one of the conditions (g0)–(g6), and the following hypotheses.*

(f3) *$f(\cdot, x)$ is measurable for each $x \in J$.*

(f4) *$f(t, x) \leq h(t)$ for all $(t, x) \in I \times J$, where $h \in L^1(I, \mathbb{R}_+)$, and $\int_{t_0}^{t_0+T} h(t) \, dt \leq R$.*

(f5) *For each $(s, z) \in [t_0, t_0 + T) \times (x_0, x_0 + R)$ there exist $\delta > 0$ and $\epsilon > 0$ such that*
$$\limsup_{y \to x-} f(t, y) \leq f(t, x) \text{ for a.e. } t \in [s, s + \delta] \text{ and all } x \in (z, z + \epsilon], \text{ and}$$
$$f(t, x) \leq \liminf_{y \to x+} f(t, y) \text{ for a.e. } t \in [s, s + \delta] \text{ and all } x \in [z, z + \epsilon).$$

Proof. The hypotheses (f0), (f3), (f4) and (f5) ensure by [1, Theorem 2.1.4] that the IVP (1) has at least one solution. The hypotheses (f1) and (f2), combined with one of the conditions (g1)–(g6), imply by Theorem 1, Proposition 1, Corollary 1 and Remarks 1 that (1) can have only one solution. This concludes the proof. □

The next existence and uniqueness theorem is a consequence of the results of Section 3 and [3, Theorem 4.1].

Theorem 3. *The initial value problem (1) has exactly one absolutely continuous solution if f satisfies the hypotheses (f0), (f1) and (f2) combined with one of the conditions (g0)–(g6), and the following hypotheses.*

(f6) $f(t, \cdot)$ *is measurable for each $x \in J$.*

(f7) $f(t, x) \leq \frac{1}{m(x)}$ *for all $t \in I$ and for a.e. $x \in J$, where $m \in L^1(J, \mathbb{R}_+)$ and $\int_{x_0}^{x_0+R} m(s)\, ds \geq T$.*

(f8) $\liminf_{s \to t-} f(s, x) \geq f(t, x)$ *for all $t \in (t_0, t_0 + T]$ and for a.e. $x \in J$ and $f(t, x) \geq \limsup_{s \to t+} f(s, x)$ for all $t \in [t_0, t_0 + T)$ and for a.e. $x \in J$.*

Proof. The uniqueness part follows from the results of Section 3, as in the proof of Theorem 2. The existence part is a consequence of [3, Theorem 4.1]. □

For the sake of completeness we shall present "classical" results, where the uniqueness conditions are complementary to those of Theorem 1, Proposition 1 and Corollary 1.

Theorem 4. *The results of Theorems 1–3 hold also when the hypotheses (f1) and (f2) are replaced by (f4) and the following hypothesis.*

(f9) $f(t, z) - f(t, y) \leq g(t, z - y)$ *for all $y, z \in J$, $y \leq z$, and for a.e. $t \in I$, where $g\colon I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies one of the following conditions.*

(g00) $u(t) \equiv 0$ *is the only absolutely continuous lower solution of the IVP*

$$u'(t) = \min\{g(t, u(t)), h(t)\} \quad \text{for a.e.} \quad t \in I, \quad u(t_0) = 0. \tag{8}$$

(g01) $g(\cdot, x)$ *is measurable for all $x \geq 0$, for each $(t, z) \in [t_0, t_0 + T) \times \mathbb{R}_+$ there exist $\delta > 0$ and $\epsilon > 0$ such that*

$\limsup_{y \to x-} g(t, y) \leq g(t, x)$ *for a.e. $t \in [s, s + \delta]$ and all $x \in (z, z + \epsilon]$, and*
$g(t, x) \leq \liminf_{y \to x-} g(t, y)$ *for a.e. $t \in [s, s + \delta]$ and all $x \in [z, z + \epsilon)$,*
and $u(t) \equiv 0$ is the only solution of the IVP (8).

(g02) $g(\cdot, x)$ *is measurable for all $x \geq 0$, $g(t, \cdot)$ is increasing or continuous for a.e. $t \in I$, and $u(t) \equiv 0$ is the only solution of the IVP (8).*

(g03) $g(t, y) = p(t)\varphi(y)$, $t \in I$, $y \geq 0$, *where $p \in L^1(I, \mathbb{R}_+)$, $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, $\varphi(y) > 0$ when $y > 0$, and $\int_{0+}^{1} \frac{dy}{\varphi(y)} = \infty$;*

(g04) $g(t, y) = \begin{cases} p(t) \max\{y \ln \frac{1}{y} \cdots \ln_n \frac{1}{y}, y \exp_n(1)\} & y > 0, \ t \in I, \\ 0, \ y = 0, \ t \in I, \end{cases}$ *where $p \in L^1(I, \mathbb{R}_+)$;*

(g05) $g(t, y) = p(t)y$, $t \in I$, $y \geq 0$, *where $p \in L^1(I, \mathbb{R}_+)$.*

As an elementary consequence of the above existence and uniqueness theorems one obtains the result presented in the Introduction.

Corollary 2. *The IVP (1) has exactly one solution if f is monotone nonincreasing or nondecreasing with respect to both of its arguments, and if $0 < f(t_0, x_0), f(t_0 + T, x_0 + R) \leq \frac{R}{T}$.*

Remarks 2. We point out that if in Corollary 2 the function f is monotone nondecreasing with respect to x and monotone nonincreasing with respect to t then uniqueness is not ensured and if f is monotone nonincreasing with respect to x and monotone nondecreasing with respect to t then existence can fail, as it is shown by the following examples: consider the functions

$$f_1(t, x) = \begin{cases} 1/2, & 0 \leq x < t, \ t \in (0, 1], \\ 1, & t \leq x \leq 1, \ t \in [0, 1], \end{cases} \quad \text{and} \quad f_2(t, x) = \begin{cases} 1, & 0 \leq x < t, \ t \in (0, 1], \\ 1/2, & t \leq x \leq 1, \ t \in [0, 1]. \end{cases}$$

It is clear that problem $x'(t) = f(t, x(t))$ a.e. in $[0, 1]$, $x(0) = 0$ with $f = f_1$ has two solutions, $x_1(t) = t/2$ and $x_2(t) = t$ for all $t \in [0, 1]$, and for $f = f_2$ the problem has not solutions.

If f in Corollary 2 is monotone nondecreasing, then assuming that $AC(I)$, is ordered pointwise, and denoting $[a, b] = \{x \in AC(I) \mid a \leq x \leq b\}$, where $a(t) \equiv x_0$ and $b(t) \equiv x_0 + R$, the equation

$$Gx(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds, \qquad t \in I \tag{9}$$

defines an increasing operator $G \colon [a, b] \to [a, b]$. Since

$$|(Gx)'(t)| \leq f(t_0 + T, x_0 + R) \quad \text{for all } x \in [a, b] \text{ and for a.e. } t \in I,$$

it follows from Proposition 1.4.4 of [6] that G has a fixed point x, which is the solution of (9). Moreover, it follows from [6, Theorem 1.1.1] that $x = \max C$, where C is the well-ordered subset of $[a, b]$ satisfying

$a = \min C$ and if $a < x \in [a, b]$, then $x \in C$ if and only if $x = \sup G(\{y \in C \mid y < x\})$. The first elements of C are iterations $G^n a$, $n = 0, 1, \ldots$, as long as this sequence is strictly increasing. If $G^n a = G^{n+1} a$ for some n, then $x = G^n a = \max C$ is the solution of the IVP (1).

If f in Corollary 2 is decreasing, then the operator G, defined by (9), is decreasing, so that G^2 is increasing. Replacing G by G^2 in the above definition of C we obtain a well-ordered subset of $[a, b]$ whose maximum is a fixed point of G^2. Since a fixed point x of G is also a fixed point of G^2, and since x is also a solution of (1), and hence uniquely determined, then $x = \max C$ is the solution of (1). In this case the sequence $(G^{2n} a)$ is increasing and bounded above by a decreasing sequence $(G^{2n+1} a)$. If $G^{2n} a = G^{2n+1} a$ for some n, then $x = G^{2n} a = \max C$ is the solution of the IVP (1). Both these cases are demonstrated by concrete examples 5 and 6 below.

4. EXAMPLES AND COUNTER-EXAMPLES

Our first example describes the importance of the hypotheses (f0) and (f1), and that they allow f to vanish at some points, including (t_0, x_0).

Example 1. The function $f \colon [0,1] \times [0,1] \to \mathbb{R}$, defined by

$$f(t,x) = ct + \frac{1}{2}x^{1/3}, \quad t, x \in [0,1],$$

satisfies the hypotheses (f0) and (f1) when $0 < c \leq \frac{1}{2}$. In fact, the hypotheses of Theorem 2 are valid, whence the IVP (1) has a unique solution when $t_0 = x_0 = 0$. Notice that $f(t_0, x_0) = 0$. But condition (f0) is no longer valid when $c = 0$. In this case (1) has an infinite number of nonnegative solutions.

The following example shows that the uniqueness condition of Theorem 1 is weaker than those of Proposition 1 and Corollary 1.

Example 2. Define $f \colon I \times J \to \mathbb{R}$ by

$$f(t,x) = \begin{cases} 2, & t = t_0 \text{ and } x = x_0, \\ 1, & t > t_0 \text{ or } x > x_0. \end{cases}$$

In this case the hypotheses (f1) and (f2) hold when $M(x) \equiv 1$ and

$$g(t,x) = \begin{cases} 1, & t = t_0 \text{ and } x = x_0, \\ 0, & t > t_0 \text{ or } x > x_0. \end{cases}$$

It is easy to see that for this function g the IVP (4) has no solutions. But condition (g0) is satisfied, whence the IVP (1) has at most one solution. In fact, the function $x(t) = x_0 + t - t_0$, $t \in I$, is the unique solution of (1).

The next example shows that uniqueness conditions of Theorem 4 don't guarantee the existence of a solution of (1).

Example 3. Let $f \colon [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(t,x) = \begin{cases} 1, & x < t, \\ \frac{1}{2}, & x \geq t, \end{cases} \quad t \in I.$$

f is positive-valued and bounded, and satisfies the hypothesis (f9) of Theorem 4 with $g(t,x) \equiv 0$, so that the IVP (1) has at most one solution. Note however that f is nondecresing in t but it is nonincreasing in x, so the assumptions of Corollary 2 are not satisfied. Moreover f satisfies neither the hypothesis (f5) nor the hypothesis (f7), whence the hypotheses of Theorem 2 or Theorem 3 are not satisfied either. It can be shown that (1) has no solutions when $x_0 = t_0 = 0$.

If uniqueness conditions given in Section 3 and in Theorem 4 don't hold, the IVP (1) may have many solutions, as shown in the next example.

Example 4. Let $f \colon [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(t,x) = \begin{cases} 1/2, & x < t, \\ 1, & x \geq t, \end{cases} \quad t \in I.$$

f is positive-valued and bounded, but does not satisfy any of the given uniqueness conditions. In this case the IVP (1) has two solutions: $x_1(t) = \frac{t}{2}$ and $x_2(t) = t$, $t \in I = [0,1]$.

Example 5. Consider the IVP

$$\begin{cases} x'(t) = \frac{1}{4}(1 + H(4t - 1)) + \frac{[2t+x(t)]}{2+2|[2t+x(t)]|} & \text{a.e. in } I = [0, 1], \\ x(0) = 1, \end{cases} \tag{10}$$

where H is the Heaviside function: $H(x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0, \end{cases}$ and $[x]$ denotes the greatest integer $\leq x$. The IVP (10) is of the form (1) with

$$f(t, x) = \frac{1}{4}(1 + H(4t - 1)) + \frac{[2t + x]}{2 + 2|[2t + x]|} \quad t \in I, \ x \in J = [1, 2]. \tag{11}$$

It is easy to see that the hypotheses of Corollary 2 hold, and that f is increasing. Thus the IVP (10) has exactly one solution. To determine it, calculate the iterations $y_n = G^n a$, $n = 0, 1, \ldots$, where $a(t) \equiv 1$ and G is defined by (9) with $t_0 = 0$, $x_0 = 1$ and f given by (11), or equivalently, successive approximations

$$\begin{cases} y'_{n+1}(t) = \frac{1}{4}(1 + H(4t - 1)) + \frac{[2t+y_n(t)]}{2+2|[2t+y_n(t)]|} & \text{a.e. in } I = [0, 1], \\ y_{n+1}(0) = 1, \ n = 0, 1, \ldots, \ y_0 = a, \end{cases}$$

(see, e.g., calculations in [1, Example 1.2.1]). It turns out that $G^3 a = G^4 a$, whence $x = G^3 a$ is a solution of (10) Its exact representation is

$$x(t) = \begin{cases} 1 + \frac{1}{2}t, & 0 \leq t < \frac{1}{4}, \\ \frac{15}{16} + \frac{3}{4}t, & \frac{1}{4} \leq t < \frac{17}{44}, \\ \frac{239}{264} + \frac{5}{6}t, & \frac{17}{44} \leq t < \frac{553}{748}, \\ \frac{5233}{5984} + \frac{7}{8}t, & \frac{553}{748} \leq t \leq 1. \end{cases}$$

Another way to determine x is a direct reasoning.

Example 6. Consider the IVP

$$\begin{cases} x'(t) = 1 - \frac{1}{4}H(4t - 1) - \frac{[2t+x(t)]}{2+2|[2t+x(t)]|} & \text{a.e. in } I = [0, 1], \\ x(0) = 1. \end{cases} \tag{12}$$

where H is the Heaviside function: $H(x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0, \end{cases}$ and $[x]$ denotes the greatest integer $\leq x$. The IVP (12) is of the form (1) with

$$f(t, x) = 1 - \frac{1}{4}H(4t - 1) - \frac{[2t + x]}{2 + 2|[2t + x]|} \quad t \in I, \ x \in J = [1, 2]. \tag{13}$$

It is easy to see that the hypotheses of Corollary 2 hold. In this case f is decreasing. The solution of (12) can be determined also in this case by calculating the iterations $y_n = G^n a$, $n = 0, 1, \ldots$, where $a(t) \equiv 1$ and G is defined by (9) with $t_0 = 0$, $x_0 = 1$ and f given by (13), or equivalently, successive approximations

$$\begin{cases} y'_{n+1}(t) = 1 - \frac{1}{4}H(4t - 1) - \frac{[2t+y_n(t)]}{2+2|[2t+y_n(t)]|} & \text{a.e. in } I = [0, 1], \\ y_{n+1}(0) = 1, \ n = 0, 1, \ldots, \ y_0 = a, \end{cases}$$

It turns out that $G^2 a = G^3 a$, whence $x = G^2 a$ is a solution of (12) Its exact representation is

$$x(t) = \begin{cases} 1 + \frac{3}{4}t, & 0 \leq t < \frac{1}{4}, \\ \frac{17}{16} + \frac{1}{2}t, & \frac{1}{4} \leq t < \frac{3}{8}, \\ \frac{35}{32} + \frac{5}{12}t, & \frac{3}{8} \leq t < \frac{183}{232}, \\ \frac{2091}{1856} + \frac{3}{8}t, & \frac{183}{232} \leq t \leq 1. \end{cases}$$

Another way also in this case to determine x is a direct reasoning.

Example 7. The numbers

$$c(n_0, \ldots, n_m) = 1 - 2^{-m-2} - \sum_{k=0}^{m} 2^{-k-m-3} \prod_{j=0}^{k} 2^{-n_j} - 2^{-2m-3} \prod_{j=0}^{m} 2^{-n_j},$$

where $m, n_0, \ldots, n_m \in \mathbb{N}$, form a well-ordered set C of rational numbers with $\min C = 1/2$ and $\sup C = 1$ (cf. [4, Ex. 1.1.1]). Define

$$\psi(z) = \begin{cases} \frac{1}{2}, & z = 0, \\ c(n_0, \ldots, n_m), & c(n_0, \ldots, n_m) - \frac{1}{2} < z < c(n_0, \ldots, n_m + 1) - \frac{1}{2}, \\ \frac{1}{2}(c(n_0, \ldots, n_m + 1) + c(n_0, \ldots, n_m)), & z = c(n_0, \ldots, n_m + 1) - \frac{1}{2}, \\ m, n_0, \ldots, n_m \in \mathbb{N}, \\ 1, & \frac{1}{2} \leq z \leq 1, \end{cases}$$

and

$$f(t, x) = \psi((t + x)/2), \quad t, x \in [0, 1].$$

It is easy to see that f satisfies the hypotheses of Corollary 2 when $I = J = [0, 1]$, so that the IVP (1) with $t_0 = x_0 = 0$ has exactly one solution.

Similarly, the function

$$f(t, x) = \frac{3}{2} - \psi((t + x)/2), \quad t, x \in [0, 1]$$

satisfies the hypotheses of Corollary 2. Thus also in this case the solution of (1) exists and is uniquely determined.

5. FINAL REMARKS

Just for completeness and for the convenience of the reader we recall how to reduce the study of other problems to the case considered in this paper.

Nonpositive nonlinearities. Let us consider the problem

$$y'(t) = g(t, y), \qquad y(t_0) = x_0, \tag{14}$$

where $g : [t_0, t_0 + T] \times [x_0 - R, x_0] \to (-\infty, 0]$, $T > 0$ and $R > 0$.

If $y : [t_0, t_0 + \alpha] \to [x_0 - R, x_0]$, $0 < \alpha < T$, is a solution of (14) then $x(t) := 2x_0 - y(t)$, $t \in [t_0, t_0 + \alpha]$ is a solution of

$$x'(t) = f(t, x), \qquad x(t_0) = x_0, \tag{15}$$

with $f(t,x) := -g(t, 2x_0 - x)$ for all $(t,x) \in [t_0, t_0 + T] \times [x_0, x_0 + R]$.

Conversely, if $x : [t_0, t_0 + \alpha] \rightarrow [x_0, x_0 + R]$, $0 < \alpha < T$, solves (15) then $y(t) := 2x_0 - x(t)$, $t \in [t_0, t_0 + \alpha]$, is a solution of (14).

Thus it is clear that the study of existence and uniqueness of solution of (14) is equivalent to that of (15), which may fall inside the scope of the present paper's results because $f \geq 0$.

Solvability on the left of t_0. Terminal value problems of the type

$$y'(t) = g(t, y), \qquad y(t_0) = x_0, \tag{16}$$

where $g : [t_0 - T, t_0] \times [x_0 - R, x_0 + R] \rightarrow \mathbb{R}$, $T > 0$ and $R > 0$, can be reduced to initial value problems. Indeed, if $y : [t_0 - \alpha, t_0] \rightarrow [x_0 - R, x_0 + R]$, $0 < \alpha < T$, is a solution of (16) then $x(t) := y(2t_0 - t)$, $t \in [t_0, t_0 + \alpha]$, is a solution of the initial value problem (15) with $f(t,x) := -g(2t_0 - t, x)$ for all $(t,x) \in [t_0, t_0 + T] \times [x_0 - R, x_0 + R]$ and, conversely, to any solution of this problem there corresponds a solution of (16).

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