Periodic solutions for second order differential equations with discontinuous restoring forces

J. Ángel Cid

Departamento de Análise Matemática,

Facultade de Matemáticas, Campus Sur,

Universidade de Santiago de Compostela,

15782 Santiago de Compostela, Spain.

E-mail : angelcid@usc.es

and

Luis Sanchez*

Centro de Matemática e Aplicações Fundamentais,

Universidade de Lisboa,

Avenida Professor Gama Pinto 2,

1649-003 Lisboa, Portugal.

E-mail : sanchez@ptmat.fc.ul.pt

1

Abstract

We deal with the existence of periodic solutions for problems with a jump discontinuity. We use an approximation procedure and the method of the lower and upper solutions.

Keywords: Periodic solutions; Discontinuous problems; Second order differential equations; Lower and upper solutions.

1   Introduction

The goal of this paper is to establish the existence of solutions of the problem

$$u'' + g(u) = h(t), \quad u(0) = u(T), \ u'(0) = u'(T). \qquad (1.1)$$

where h is continuous in $[0, T]$ and g is continuous in $\mathbb{R} \setminus \{0\}$ with a jump discontinuity at $u = 0$.

Several authors have dealt with boundary value problems involving discontinuous functions, using the methods of Nonlinear Analysis suitable to obtain existence results in the theory of ordinary and partial differential equations. Let us mention that problems concerning elliptic equations have been tackled in [1] using a dual action variational technique; similar problems have been considered in [3] by means of critical point theory for locally Lipschitz functionals. We shall use an approximation procedure in which g is replaced by a sequence of continuous functions that "fill the gap" between $g(0^-)$ and $g(0^+)$; this method has been already used for instance in [5] for elliptic problems and in [4] for periodic problems with dry friction.

Of course, there is a rich literature in the field of differential inclusions, and in recent years a lot of attention has been given to the periodic boundary value problem for inclusions of the first and second order. Topological and variational methods have been developed to extend to differential inclusions some significant existence results in the area of ordinary differential equations. Only to mention some recent work, we refer the reader

2

to [2, 9, 11, 12, 15] and their references , where research in this field may be traced back. The discontinuities in the right-hand sides considered by these authors are, of course, much more general than ours. However, particular features of the asymptotic behaviour of g as the ones that interest here do not seem to have been covered in the literature. Moreover, less attention has been given to multiplicity of solutions.

On the other hand, problem (1.1) with g continuous (which is our starting point) has inspired a huge amount of work: in the last quarter of century significant steps have been given towards the understanding of existence, multiplicity and properties of its solutions. These depend, of course, on which type of restoring term g one is interested in.

We shall consider mainly two types of behaviour for g:

1) g is positive everywhere and vanishes at ∞;

2) roughly speaking, g takes values above and below the mean value $\bar{h}$ of $h(t)$ and has its growth linearly restricted on one side.

The first type has been recently studied by Ward [18]. We improve his existence results by adding multiplicity and we show that multiplicity persists in the discontinuous case.

The second type has been studied by many authors. In connection with our approach, we should mention that two important devices that have been used to deal with this type of forces are a Landesman-Lazer condition (see [10]) and the sign condition $u(g(u) - \bar{h}) \geq 0$ for large $|u|$ (see [7, 14, 16, 17]). These and related situations have been approached by means of topological and variational methods, the use of upper and lower solutions included. Very recently, De Coster and Tarallo [6] have revisited the problem and introduced new techniques. In fact we make an intensive use of one of their results here: an important tool in our arguments consists in obtaining a solution of (1.1) provided that g is bounded and a lower solution and an upper one are known, independently of any ordering between them (cf. [6]). See theorem A below. This provides simpler proofs than the classical ones in similar situations.

The nonlinearities of the second type that we consider in this paper are such that $g(u) - \bar{h}$ may change sign in arbitrary large intervals. Also, we never use the assumption that the potential (the primitive G of $g - \bar{h}$) is coercive. In this way our results, even in the continuous case, do not seem to be contained in the above mentioned literature.

We would like to underline that, while working with the approximation procedure, we need to be able to localize the solutions of the intermediate problems. Hence we rely on classical techniques to deal with existence of solutions but we always care to exhibit an explicit bound for those solutions. This is done by using theorem A in combination with some features of the problem under analysis.

It will be clear in the outset that the same method would allow us to obtain analogous results for functions $g(u)$ with finitely many jump discontinuities.

2 Bounded and positive nonlinearities

In our work the following result, which is a particular case of theorem 3.4 in [6], is fundamental.

Theorem A *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, $h \in L^1(0, T)$ and let there exist α, $\beta \in H_T^1$ lower and upper solutions, respectively, of (1.1) with $\alpha \quad \beta$ (H_T^1 denotes the Sobolev space of $T-$periodic funtions). Assume moreover there exists $M \geq 0$ such that for all $s \in \mathbb{R}$*

$$|g(s)| \leq M.$$

Then the problem (1.1) has at least one solution $u \in W^{2,1}(0, T)$ with $u \in \mathcal{S}$ where

$$\mathcal{S} = \{u \in \mathcal{C}([0, T]) : \exists t_1, t_2 \in [0, T], u(t_1) \geq \beta(t_1), \ \alpha(t_2) \geq u(t_2)\}. \quad (2.2)$$

4

2.1 The continuous case

In [18] the author deals with problem (1.1) under the following condition:

(G1) Assume $g(s) > 0$ for all s and $g(-\infty) = g(+\infty) = 0$.

For any function $w \in L^1(0, T)$ let

$$\bar{w} = \frac{1}{T} \int_0^T w(s)ds \quad \text{and} \quad \tilde{w} = w - \bar{w},$$

and for $g : \mathbb{R} \to \mathbb{R}$ bounded let $\|g\|_\infty = \sup_{s \in \mathbb{R}} |g(s)|$.

We introduce the space $\tilde{\mathcal{C}}([0, T])$ whose elements are the T-periodic continuous real functions with zero mean value in $[0, T]$. As is well known any continuous, T-periodic real function $u(t)$ splits as $u(t) = \bar{u} + \tilde{u}(t)$, where $\bar{u}$ is the mean value of u and $\tilde{u} \in \tilde{\mathcal{C}}([0, T])$.

The following theorem is the main result of [18] (theorem 1 in [18]).

Theorem B *Let* $g \in \mathcal{C}(\mathbb{R})$ *satisfy* (G1). *Then for* $\tilde{h} \in \tilde{\mathcal{C}}([0, T])$ *there is a number* $\lambda^* = \lambda^*(\tilde{h})$ *satisfying* $0 < \lambda^*(\tilde{h}) \leq \|g\|_\infty$ *such that the periodic problem (1.1) with* $h(t) = \bar{h} + \tilde{h}(t)$ *has a solution if and only if* $0 < \bar{h} \leq \lambda^*(\tilde{h})$.

Remark 2.1 *In theorem 1 of [18] the following condition is also imposed*

(G2) *Let* $G(s) = \int_0^s g(t)dt$, *and assume there is a number* $M \geq 0$ *such that* $|G(s)| \leq M$ *for all* $s \in \mathbb{R}$,

but a carefully reading of the proof shows that (G2) is not needed.

In the following proposition we are going to prove the existence of lower and upper solutions for problem (1.1).

Proposition 2.1 *Let* $g \in \mathcal{C}(\mathbb{R})$ *satisfy* (G1). *Then for* $\tilde{h} \in \tilde{\mathcal{C}}([0, T])$ *and* $0 < \bar{h} < \lambda^*(\tilde{h})$ *there exist a strict lower solution* α *and two upper solutions* β_1, β_2 *of problem (1.1) with* $h(t) = \bar{h} + \tilde{h}(t)$, *such that*

$$\beta_1(t) \leq \alpha(t) \leq \beta_2(t) \quad \text{for all} \quad t \in [0, T].$$

Proof. Choose $\lambda \in (\bar{h}, \lambda^*(\tilde{h}))$. The problem

$$u'' + g(u) = \tilde{h}(t) + \lambda, \quad u(0) = u(T), \ u'(0) = u'(T),$$

has, by theorem B, a solution α which is a strict lower solution for (1.1).

Now, let w be the unique periodic solution of $u'' = \tilde{h}(t)$ with mean value zero. Since $g(-\infty) = g(+\infty) = 0$, there exists $c > 0$ such that $\beta_1 = w - c$ and $\beta_2 = w + c$ are upper solutions for (1.1) and moreover

$$\beta_1(t) \le \alpha(t) \le \beta_2(t) \quad \text{for all} \quad t \in [0, T].$$

$\square$

As a direct consequence of proposition 2.1 and theorem A we have the following multiplicity result, which improves theorem B.

Theorem 2.2 *Let $g \in \mathcal{C}(\mathbb{R})$ satisfy (G1). Then for $\tilde{h} \in \tilde{\mathcal{C}}([0, T])$ there is a number $\lambda^* = \lambda^*(\tilde{h})$ satisfying $0 < \lambda^*(\tilde{h}) \le \|g\|_\infty$ such that the periodic problem (1.1) with $h(t) = \bar{h} + \tilde{h}(t)$*

i) has at least two solutions if $0 < \bar{h} < \lambda^*(\tilde{h})$,

ii) has at least one solution if $\bar{h} = \lambda^*(\tilde{h})$,

iii) has no solution if $\bar{h} \notin (0, \lambda^*(\tilde{h})]$.

Remark 2.2 *Using phase plane analysis it is easy to give examples of problems of the form*

$$u'' + g(u) = \bar{h}, \quad u(0) = u(T), \ u'(0) = u'(T),$$

where g satisfies (G1), is strictly increasing in $(-\infty, 0]$ and strictly decreasing in $[0, +\infty)$ and whose only solutions for a small enough period $T > 0$ are constant. Hence the above result cannot be in general improved.

Next we prove that λ^* is an increasing function of g.

Lemma 2.3 *Let $g_1, g_2 \in \mathcal{C}(\mathbb{R})$ satisfy (G1) and such that $g_1(s) \le g_2(s)$ for all $s \in \mathbb{R}$. Then for $\tilde{h} \in \tilde{\mathcal{C}}([0, T])$ we have, with obvious notation,*

$$\lambda^*(g_1, \tilde{h}) \le \lambda^*(g_2, \tilde{h}).$$

6

Proof. By theorem B problem

$$u'' + g_1(u) = \tilde{h}(t) + \lambda^*(g_1, \tilde{h}), \quad u(0) = u(T), \ u'(0) = u'(T),$$

has a solution α, which is a lower solution for

$$u'' + g_2(u) = \tilde{h}(t) + \lambda^*(g_1, \tilde{h}), \quad u(0) = u(T), \ u'(0) = u'(T), \qquad (2.3)$$

because $g_1(s) \leq g_2(s)$ for all $s \in \mathbb{R}$.

Repeating the argument of proposition 2.1 we have that there exists an upper solution β for (2.3) with $\alpha \leq \beta$. Therefore, problem (2.3) has a solution and by theorem B we deduce that $\lambda^*(g_1, \tilde{h}) \leq \lambda^*(g_2, \tilde{h})$. $\qquad \square$

In theorem 2 of [18] the author points out that in most cases $\lambda^*(\tilde{h}) < \|g\|_\infty$. We are going to give a sufficient and necessary condition for the equality $\lambda^*(\tilde{h}) = \|g\|_\infty$ to hold.

Proposition 2.4 *Let $g \in \mathcal{C}(\mathbb{R})$ satisfy* (G1) *and be $\tilde{h} \in \tilde{\mathcal{C}}([0, T])$. Then $\lambda^*(\tilde{h}) = \|g\|_\infty$ if and only if there exists an interval (maybe degenerate) $I \subset g^{-1}(\|g\|_\infty)$ such that $l(v([0, T])) \leq l(I)$, where v is the unique solution with mean value zero of problem*

$$u'' = \tilde{h}(t), \quad u(0) = u(T), \ u'(0) = u'(T), \qquad (2.4)$$

and l denotes length.

Proof. If $\lambda^*(\tilde{h}) = \|g\|_\infty$ then by theorem B there exists a solution v_1 of problem

$$u'' + g(u) = \tilde{h}(t) + \|g\|_\infty, \quad u(0) = u(T), \ u'(0) = u'(T). \qquad (2.5)$$

Integrating the equation over $[0, T]$ and using the periodicity conditions we have that

$$\frac{1}{T} \int_0^T g(v_1(s))ds = \|g\|_\infty,$$

or equivalently

$$\frac{1}{T} \int_0^T (\|g\|_\infty - g(v_1(s)))ds = 0,$$

and since $\|g\|_\infty \geq g(v_1(s))$ for all $s \in [0, T]$ we have that

$$g(v_1(s)) = \|g\|_\infty \quad \text{for all} \quad s \in [0, T].$$

Therefore v_1 is a solution of (2.4) and then we have that $v_1 = v + c$. Moreover, since $g(v_1(s)) = \|g\|_\infty$ for all $s \in [0, T]$, we have that there exists an interval $I \subset g^{-1}(\|g\|_\infty)$ such that $l(v([0, T])) = l(v_1([0, T])) \leq l(I)$.

Conversely, if there exists $I \subset g^{-1}(\|g\|_\infty)$ such that $l(v([0, T])) \leq l(I)$, where v is the unique solution with mean value zero of (2.4), we can choose $c \in \mathbb{R}$ such that $v_1 = v + c$ satisfies $v_1([0, T]) \subset I$ and then v_1 is a solution of (2.5). Thus we deduce that $\lambda^*(\tilde{h}) = \|g\|_\infty$. □

Now we are in a position to give some estimates for λ^*.

Proposition 2.5 *Let $g \in \mathcal{C}(\mathbb{R})$ satisfy* (G1) *and be $\tilde{h} \in \tilde{\mathcal{C}}([0, T])$. If for some $\lambda \in (0, \|g\|_\infty]$ there exists an interval $I \subset g^{-1}([\lambda, \infty))$ such that $l(v([0, T])) \leq l(I)$, where v is the unique solution with mean value zero of (2.4), then we have that*

$$0 < \lambda \leq \lambda^*(\tilde{h}) \leq \|g\|_\infty.$$

Proof. We consider the truncated function $g_\lambda(s) = g(s)$ if $g(s) \leq \lambda$ and $g_\lambda(s) = \lambda$ if $g(s) \geq \lambda$. It is obvious that $g_\lambda(s) \leq g(s)$ for all $s \in \mathbb{R}$ and that $\|g_\lambda\|_\infty = \lambda$. The hypotheses and proposition 2.4 imply that $\lambda^*(g_\lambda, \tilde{h}) = \lambda$ and then by lemma 2.3 we deduce that $\lambda \leq \lambda^*(\tilde{h})$. □

2.2 The discontinuous case

In this section we are going deal with problem (1.1) considering a function g with a jump discontinuity. Our assumptions on g are (G1) and

(D1) $g : \mathbb{R} \to \mathbb{R}$ is a continuous function in $\mathbb{R} \setminus \{0\}$ and the limits

$$g(0^\pm) = \lim_{x \to 0^\pm} g(x) > 0$$

exist and are finite.

We notice that (G1) and (D1) imply that g is bounded.

Definition 2.1 *For g satisfying conditions* (G1) *and* (D1) *and for each $h \in \mathcal{C}([0, T])$ we mean by a generalized solution of problem (1.1) a function*

$u \in W^{2,1}(0, T)$ with $u(0) = u(T)$, $u'(0) = u'(T)$ and such that there exists $w : [0, T] \to \mathbb{R}$ satisfying

i) $u''(t) + w(t) = h(t)$ a.e. $t \in [0, T]$;

ii) $w(t)$ belongs to the interval with end points $g(0^-)$ and $g(0^+)$ for a.e.
$t \in \Omega := \{t \in [0, T] : u(t) = 0\}$; and

iii) $w(t) = g(u(t))$ for a.e. $t \in [0, T] \setminus \Omega$.

It should be remarked that, depending on the "gap" of g at the origin and the amplitude of the oscillation $h(t)$, problem (1.1) may have a *trivial generalized solution*, namely the constant zero, if the range of $h(t)$ is contained in the interval whose endpoints are $g(0^-)$ and $g(0^+)$. For a given g, provided that $h(t)$ oscillates enough, this phenomenon cannot occur, so that our existence results do not reduce to trivial statements.

Next, we present the main result of this section about the multiplicity of generalized solutions for the periodic problem (1.1).

Theorem 2.6 *Assume that* (G1) *and* (D1) *hold. For each* $\tilde{h} \in \tilde{\mathcal{C}}([0, T])$ *there exists* $0 < \hat{\lambda}(\tilde{h}) \leq \|g\|_\infty$ *such that if* $0 < \bar{h} < \hat{\lambda}(\tilde{h})$ *then problem* *(1.1) with* $h(t) = \tilde{h}(t) + \bar{h}$ *has at least two generalized solutions.*

Proof.   *1).- Approximated problems*

It is easy to see that there exist a sequence of positive numbers $t_n \downarrow 0$ and a nondecreasing sequence of continuous functions $\{g_n\}_{n=1}^\infty$ with the following properties:

$$g_n(s) = g(s) \quad \text{if} \quad |s| \geq t_n, \tag{2.6}$$

and, given $m \in \mathbb{N}$ there exists $\delta > 0$ such that (in case $g(0^-) < g(0^+)$; otherwise the following inequalities are reversed)

$$g(0^-) - \frac{1}{m} \leq g_n(s) \leq g(0^+) + \frac{1}{m} \quad \text{if } |s| \leq \delta \text{ and for all } n \in \mathbb{N}. \tag{2.7}$$

9

In fact, it suffices to set (in case $g(0^-) < g(0^+)$; the procedure in the other case is analogous)

$$g_n(s) = \begin{cases} g(s) & \text{if} \quad s < 0 \quad \text{or} \quad s > t_n, \\ g(0^-) & \text{if} \quad s = 0, \\ l_n(s) & \text{if} \quad 0 \le s \le t_n, \end{cases}$$

where $l_n(t) = g(0^-) + nt$ and $t_n = \inf\{t > 0 : l_n(t) = g(t)\}$.

If we take $\hat{\lambda}(\tilde{h}) = \lambda^*(g_1, \tilde{h})$ by lemma 2.3 we have that $\hat{\lambda}(\tilde{h}) \le \lambda^*(g_n, \tilde{h})$ for all $n \in \mathbb{N}$.

Fix $\bar{h}$ such that $0 < \bar{h} < \hat{\lambda}(\tilde{h})$ and consider for all $n \in \mathbb{N}$ the approximated problems (P_n) with the continuous function g_n

$$(P_n) \quad u'' + g_n(u) = \tilde{h}(t) + \bar{h}, \quad u(0) = u(T), \ u'(0) = u'(T).$$

Take now $\bar{h} < \bar{\bar{h}} < \hat{\lambda}(\tilde{h})$ and let α be the solution of

$$u'' + g_1(u) = \tilde{h}(t) + \bar{\bar{h}}, \quad u(0) = u(T), \ u'(0) = u'(T),$$

which exists by theorem B. Since $\{g_n\}_{n=1}^{\infty}$ is nondecreasing α is a strict lower solution of (P_n) for all $n \in \mathbb{N}$. Using a similar argument to that of the proof of proposition 2.1 we obtain a pair of upper solutions β_1 and β_2 for all problems (P_n) such that

$$\beta_1(t) \le \alpha(t) \le \beta_2(t) \quad \text{for all } t \in [0, T].$$

Then, there exists a solution v_1 of (P_1) such that

$$\alpha(t) < v_1(t) \le \beta_2(t) \quad \text{for all } t \in [0, T].$$

We notice that the symbol "$<$" appears instead of "$\le$" because α is a strict lower solution. Now v_1 is a lower solution for (P_2) and repeating the process we have a nondecreasing sequence

$$\alpha < v_1 \le v_2 \le \ldots \le v_n \le v_{n+1} \le \ldots \le \beta_2, \qquad (2.8)$$

where v_n is a solution of (P_n).

On the other hand, by theorem A, for each $n \in \mathbb{N}$ there exists a solution u_n of problem (P_n) such that for some $t_n^1, t_n^2 \in [0, T]$ we have that

$$u_n(t_n^1) \geq \beta_1(t_n^1) \quad \text{and} \quad u_n(t_n^2) \leq \alpha(t_n^2). \tag{2.9}$$

2).- Passing to the limit.

By assumptions (G1) and (D1), by property (2.6) and by the fact that $\{g_n\}_{n=1}^{\infty}$ is nondecreasing there exists $M_1 > 0$ such that $\|g_n\|_\infty \leq M_1$ for all $n \in \mathbb{N}$. Since v_n is a solution of (P_n) we deduce that there exists $M_2 > 0$ such that $\|v_n''\|_\infty \leq M_2$ for all $n \in \mathbb{N}$. Moreover $\{v_n\}_{n=1}^{\infty}$ is bounded and then there exists $M_3 > 0$ such that $\|v_n'\|_\infty \leq M_3$ for all $n \in \mathbb{N}$. Therefore, Ascoli's theorem and (2.8) imply that $\{v_n\}_{n=1}^{\infty}$ converges uniformly (even in the $\mathcal{C}^1$ norm) to a continuous function v which satisfies $v(0) = v(T)$ and $v'(0) = v'(T)$.

On the other hand, $\{v_n''\}_{n=1}^{\infty} \rightharpoonup z$ in $L^2(0,T)$ and $\{g_n(v_n)\}_{n=1}^{\infty} \rightharpoonup w$ in $L^2(0,T)$, because $\{v_n''\}_{n=1}^{\infty}$ and $\{g_n(v_n)\}_{n=1}^{\infty}$ are bounded (if it is necessary take a convergent subsequence). In particular we have that $z = v''$ in the sense of distributions and then $v \in W^{2,1}(0,T)$. Passing to the limit we obtain

$$v''(t) + w(t) = \tilde{h}(t) + \bar{h} \quad \text{for a.e. } t \in [0, T].$$

Let $K \subset [0, T] \setminus \Omega$ be a compact set, where $\Omega = \{t \in [0, T] : v(t) = 0\}$. Since $v_n \to v$ uniformly we have that $|v_n(t)| \geq c > 0$ for all $t \in K$ and for n large enough. Then by (2.6) we have that $g_n(v_n) \to g(v)$ uniformly in K and therefore

$$w(t) = g(v(t)) \quad \text{for a.e. } t \in [0, T] \setminus \Omega.$$

Now suppose for definiteness that $g(0^-) < g(0^+)$; the argument in the other case being analogous. Given $m \in \mathbb{N}$ let $\delta > 0$ be as in (2.7). Since $v_n \to v$ uniformly, there exists a $N_0 \in \mathbb{N}$ such that $|v_n(t)| \leq \delta$ in Ω for all $n \geq N_0$. Then by (2.7) we have for all $n \geq N_0$

$$g(0^-) - \frac{1}{m} \leq g_n(v_n(t)) \leq g(0^+) + \frac{1}{m} \quad \text{for all } t \in \Omega.$$

Since the set $\{x \in L^2(0, T) : g(0^-) - \frac{1}{m} \le x \le g(0^+) + \frac{1}{m}\}$ is closed in the weak topology of $L^2(0, T)$ passing to the limit we obtain

$$g(0^-) - \frac{1}{m} \le w(t) \le g(0^+) + \frac{1}{m} \quad \text{for a.e. } t \in \Omega.$$

Then $g(0^-) \le w(t) \le g(0^+)$ for a.e. $t \in \Omega$, and thus v is a generalized solution of (1.1).

By a quite similar reasoning we obtain that $\{u_n\}_{n=1}^\infty$ converges uniformly (taking a subsequence if it is necessary) to a generalized solution u of problem (1.1).

3.- There are two different solutions.

By (2.8) we have that $\alpha(t) < v_1(t) \le v(t)$ for all $t \in [0, T]$. On the other hand (2.9) implies there exists $s \in [0, T]$ such that $u(s) \le \alpha(s)$. Then $u \neq v$, and the proof is complete. $\qquad\square$

3 One-sided sublinear nonlinearities

The restoring terms considered in this section are, roughly speaking, unbounded and become larger at $+\infty$ than at $-\infty$. Our results are related to those of [7, 8, 14, 16, 17].

We shall start from a hypothesis that allows the construction of upper and lower solutions. Namely, we shall assume throughout in this section the following condition relating g and $\bar{h}$:

(G2) $g : \mathbb{R} \to \mathbb{R}$ and for each $r > 0$ there exist intervals I_r, J_r such that
$l(I_r) > r$, $l(J_r) > r$ and

$$g|_{I_r} \le \bar{h} \le g|_{J_r}.$$

Then it is clear that, choosing such intervals with sufficiently large length, we can construct an upper (respectively lower) solution $c + w(t)$ of (1.1), taking values in I (respectively J), by adding a constant c to w, the unique periodic solution of $w'' = \tilde{h}(t)$ with mean value zero. If $\sup J \le \inf I$ the lower and upper solutions are well ordered and it is well

known that a solution exists between them. Therefore in the theorems below we shall always deal with the assumption

$(\overline{G2})$ $g : \mathbb{R} \to \mathbb{R}$ and for each $r > 0$ there exist intervals I_r, J_r such that $l(I_r) > r$, $l(J_r) > r$, $\sup I_r < 0 < \inf J_r$ and

$$g|_{I_r} \le \bar{h} \le g|_{J_r}.$$

3.1 The continuous case

We present a simple existence principle.

Proposition 3.1 *Let* $g \in \mathcal{C}(\mathbb{R})$ *and suppose that in addition to* $(\overline{G2})$ *g is bounded above or is bounded below. Then for each* $\tilde{h} \in \tilde{\mathcal{C}}([0, T])$ *and* $h(t) = \bar{h} + \tilde{h}(t)$ *there exists a solution of (1.1).*

Proof. According to the above remark we can fix a lower solution α and an upper solution β of (1.1) and $\beta < 0 < \alpha$. Let the closed interval $[m, M]$ contain the range of both α and β. For definiteness assume that g is bounded below (if g is bounded above the proof is analogous), set $K := \sup_{u \in \mathbb{R}}[-g(u)] + \|h\|_\infty$, $L := \max\{|m|, M\} + KT^2$ and define a new function by setting

$$g_L(u) = \begin{cases} g(u) & \text{if} \quad |u| \le L, \\ g(-L) & \text{if} \quad u < -L, \\ g(L) & \text{if} \quad u > L. \end{cases}$$

Then consider the modified problem

$$u'' + g_L(u) = h(t), \quad u(0) = u(T), \ u'(0) = u'(T). \tag{3.10}$$

Since $g = g_L$ in $[-L, L]$ the upper and lower solutions β and α are upper and lower solutions of (3.10). Now theorem A is applicable to (3.10) and we assert that (3.10) has a solution $u(t)$ in the set $\mathcal{S}$ given in (2.2). Hence there exists $t_1 \in [0, T]$ such that $u(t_1) \in [m, M]$. On the other hand

$$\int_0^T u''^+(s)\, ds = \int_0^T (-g(u(s)) + h(s))^+\, ds \le KT,$$

13

and since due to periodicity $\int_0^T u''^+(s)\,ds = \int_0^T u''^-(s)\,ds$ and moreover, letting $t_0 \in [0,T]$ be such that $u(t_0) = \min_{t \in [0,T]} u(t)$,

$$-\int_{t_0}^t u''^-(s)ds \le u'(t) \le \int_{t_0}^t u''^+(s)ds,$$

we conclude $\|u'\|_\infty \le KT$. Then it follows that

$$|u(t)| \le |u(t_1)| + \int_{t_1}^t u'(s)\,ds \ \le \max\{|m|, M\} + KT^2 = L \quad \text{for all} \quad t \in [0,T].$$

Then, by the construction of g_L, $u(t)$ is a solution of (1.1). $\qquad\qquad\square$

In order to simplify the statement of our next theorems let us introduce the following definition. Given a function $\hat{g}$ defined in $\mathbb{R}$ we shall say that another function g is *admissible* (with respect to $\hat{g}$) if $g = \hat{g}$ in $\mathbb{R} \setminus [-1,1]$ and $\sup_{x \in [-1,1]} g(x) = \sup_{x \in [-1,1]} \hat{g}(x)$ and $\inf_{x \in [-1,1]} g(x) = \inf_{x \in [-1,1]} \hat{g}(x)$.

Theorem 3.2 *Let $\hat{g} \in \mathcal{C}(\mathbb{R})$ satisfying $(\overline{G2})$ and*

$$\limsup_{u \to -\infty} \frac{\hat{g}(u)}{u} < \frac{\pi^2}{T^2}. \tag{3.11}$$

Assume in addition that either

(i) *$\hat{g}$ is bounded below in $[0, +\infty)$, or*

(ii) *$\hat{G}(u) := \int_0^u \hat{g}(s)\,ds - \bar{h}u$ is bounded below in $[0, +\infty)$.*

Then, there exist constants c, C with the property that, for all continuous admissible functions g, problem (1.1) has a solution $u(t)$ with $c \le u(t) \le C$ for all $t \in [0,T]$.

Proof. Fix a lower solution α and an upper solution β of (1.1) as in the preceeding proof, $\limsup_{u \to -\infty} \frac{\hat{g}(u)}{u} < \nu < \frac{\pi^2}{T^2}$ and $k > 0$ such that

$$\hat{g}(u) \ge \nu u - k, \qquad \text{for all} \quad u \le 0. \tag{3.12}$$

Of course, we can suppose that any continuous admissible g satisfies the same inequality.

Claim: There exist constants c, C with the property that, for all continuous admissible functions g, any periodic solution $u(t)$ of problem (1.1) for which $u(t_1) \in [m, M]$ for some t_1 satisfies $c \le u(t) \le C$ for all $t \in [0,T]$. (Here m and M are the same used in the proof of Proposition 3.1)

14

Proof of the Claim: If $u(t)$ takes values less than m, let $[a, b]$ be an interval such that $b - a < T$, $u(t) < m$ if $a < t < b$ and $u(a) = m = u(b)$. Multiplying (1.1) by $u(t) - m$ and integrating in $[a, b]$, using (3.12), we obtain, with $k_1 = \nu|m| + k + \|h\|_\infty$:

$$\int_a^b u'(s)^2 \, ds \leq \nu \int_a^b (u(s) - m)^2 \, ds + k_1 \int_a^b |u(s) - m| \, ds.$$

Now, since $\nu < \frac{\pi^2}{T^2}$ and, by the well known Poincaré inequality (see [13, section 1.3]),

$$\frac{\pi^2}{(b-a)^2} \int_a^b (u(s) - m)^2 \, ds \leq \int_a^b u'(s)^2 \, ds,$$

we infer that there exists $k_2 = k_2(k_1, \nu)$ such that

$$\int_a^b u'(s)^2 \, ds \leq k_2.$$

Let $m_0 = u(t_0)$ be the minimum of u ($t_0 \in (a, b)$). Since $u(t_0) = u(a) + \int_a^{t_0} u'(s) ds$, it follows easily that $m_0 \geq m - (Tk_2)^{1/2}$.

Assume (i) holds. As g is bounded below in $[m - (Tk_2)^{1/2}, +\infty)$ and the lower bound can be taken to be the same for all admissible functions g, it turns out, as in the proof of proposition 3.1, that for some constant C that we can express in terms of m, M, k_2, T and $\inf_{[m-(Tk_2)^{1/2}, +\infty)} g$, $M_0 := \max u \leq C$.

Assume (ii) holds. First we note that we can estimate $u'(t)$ for $u(t) \leq M$, since for each such t there exists an interval $[t, s]$ (or $[s, t]$) with $u'(s) = 0$ and $m - (Tk_2)^{1/2} \leq u(\tau) \leq M$ for all $\tau \in [t, s]$ (or $[s, t]$). Integrating between t and s we infer

$$\sup_{u(t) \leq M} |u'(t)| \leq T(\sup_{[m-(Tk_1)^{1/2}, M]} |g(u)| + \|h\|_\infty).$$

Now let $t \in [0, T]$ be arbitrary. Multiplying (1.1) by u' and integrating in some interval $[t_1, t]$, where $u(t_1) \leq M$, we derive (with $H := \|\tilde{h}\|_\infty$)

$$\frac{u'(t)^2}{2} + G(u(t)) \leq \frac{u'(t_1)^2}{2} + G(u(t_1)) + 2H \int_{t_1}^t |u'(s)| ds.$$

Note that G is bounded below in $[m, \infty)$ by the same constant for all admissible g. In view of the preceding estimate and (ii) there exists a

15

constant $A > 0$ (the same for every function g) such that

$$u'(t)^2 \leq A^2 + 2H \int_{t_1}^{t} |u'(s)| ds.$$

From this differential inequality we obtain

$$|u'(t)| \leq A + HT \quad \text{for all} \quad t \in [0, T].$$

Hence, clearly if $M_0 = \max u$, we have $M_0 \leq M + T(A + HT)$. So that in this case we are also able to determine c and C with the properties stated. This proves the *Claim*.

Now it is straightforward to finish the proof of the theorem: it suffices to consider a modified problem, by setting g constant in each of the intervals $(-\infty, c]$ and $[C, +\infty)$. (We may assume that C is large enough so that $g(C) \geq \bar{h}$.) Then invoking Theorem A and the estimates established in the *Claim* (since the same assumptions and bounds hold for truncated functions), the result follows. □

Theorem 3.3 *Let $\hat{g} \in \mathcal{C}(\mathbb{R})$ satisfying $(\overline{G2})$, and*

$$\liminf_{u \to -\infty} \frac{\min_{u \leq s \leq 0} \hat{g}(s)}{u} < \frac{2\pi}{T^2}. \tag{3.13}$$

Assume in addition that either

(i) $\hat{g}$ is bounded below in $[0, +\infty)$, or

(ii) $\hat{G}(u) := \int_0^u \hat{g}(s) ds - \bar{h}u$ is bounded below in $[0, +\infty)$.

Then there exist constants c, C with the property that, for all continuous admissible functions g, problem (1.1) has a solution $u(t)$ with $c \leq u(t) \leq C$ for all $t \in [0, T]$.

Proof. Fix α, β, m, M as in the proceeding case. For all admissible functions g the primitives $G(u) = \int_0^u g(s) ds - \bar{h}u$ have a common lower bound. Consider the truncated function

$$g_{R,S}(u) = \begin{cases} g(u) & \text{if} \quad -R \leq u \leq S \\ g(-R) & \text{if} \quad u < -R, \\ g(S) & \text{if} \quad u > S. \end{cases}$$

16

Let $S \geq M$ be such that $g(S) \geq \bar{h}$ and, since $\liminf\limits_{u \to -\infty} \dfrac{\min_{u \leq s \leq 0} \hat{g}(s)}{u} < \dfrac{2\pi}{T^2}$,

we can pick up $R > 0$ such that, independently of S and the admissible function g, the following hold:

1) $g_{R,S}(u) \geq -\nu R$ for all $u \leq 0$, with $\nu < \frac{2\pi}{T^2}$;

2) $m - \frac{T^2}{2\pi}(\nu R + H) \geq -R$, $(H = \|h\|_\infty)$.

By Theorem A, the problem (1.1) with the function $g_{R,S}$ has a solution $u_R \in \mathcal{S}$. For simplicity we write $u = u_R$.

If $u(t)$ takes values less than m, let $[a, b]$ be an interval such that $b - a < T$, $u(t) < m$ if $a < t < b$ and $u(a) = m = u(b)$. Multiplying (1.1) by $u(t) - m$ and integrating in $[a, b]$, using (1), we obtain:

$$\int_a^b u'(t)^2 \, dt \leq (\nu R + H) \int_a^b |u(t) - m| \, dt.$$

On the other hand, by the Poincaré inequality

$$\frac{\pi^2}{(b-a)^2} \int_a^b (u(t) - m)^2 \, dt \leq \int_a^b u'(t)^2 \, dt.$$

Then, we have

$$\frac{\pi^2}{T^3} \left[\int_a^b |u(t) - m| \, dt \right]^2 \leq \frac{\pi^2}{(b-a)^2} \int_a^b (u(t) - m)^2 \, dt$$
$$\leq (\nu R + H) \int_a^b |u(t) - m| \, dt.$$

Therefore, it follows that

$$\int_a^b |u(t) - m| \, dt \leq \frac{T^3}{\pi^2}(\nu R + H),$$

and thus

$$\int_a^b u'(t)^2 \, dt \leq \frac{T^3}{\pi^2}(\nu R + H)^2.$$

Let $t_0 \in [a, b]$ such that $u(t_0) = \min\{u(t)\} := m_0$. Then

$$m - m_0 = \int_{t_0}^b u'(s)ds \leq \int_{t_0}^b u'^+(s)ds \leq \int_a^b u'^+(s)ds.$$

Since $\int_a^b u'(s)ds = 0$ we have that $\int_a^b u'^+(s)ds = \int_a^b u'^-(s)ds$ and thus $\int_a^b |u'(s)|ds = 2 \int_a^b u'^+(s)ds$. Therefore

$$m - m_0 \leq \frac{1}{2} \int_a^b |u'(s)|ds \leq \frac{1}{2}\sqrt{T \int_a^b u'(s)^2 ds}.$$

Hence $m_0 \geq m - \frac{T^2}{2\pi}(\nu R + H)$, and by 2) we have $m_0 \geq -R$.

Now, proceeding as in the proof of theorem 3.2 we deduce that, for both cases i) and ii), there exists a constant C, independent of S, such that $M_0 := \max u \leq C$. Then if we choose $S \geq \max\{C, M\}$ such that $g(S) \geq \bar{h}$ we have that

$$-R \leq u(t) \leq S \quad \text{for all } t \in [0, T],$$

and therefore $u(t)$ is a solution of problem (1.1) which moreover satisfies

$$c \leq u(t) \leq C \text{ for all } t \in [0, T],$$

with $c = -R$ and independently of the admissible function g. $\qquad\square$

Remark 3.1 *i) Suppose that $(\overline{G2})$ is rephrased simply as*

(G2'): $g : \mathbb{R} \to \mathbb{R}$ and there exist intervals I, J such that $\sup I < 0 < \inf J$ and

$$g|_I \leq \bar{h} \leq g|_J.$$

Observing that there exists a constant c such that for the solution w of $w'' = \tilde{h}(t)$ with mean value zero we have

$$\|w\|_\infty \leq c\|\tilde{h}\|_{L^1(0,T)}$$

one easily sees that the above theorems still hold with (G2)' instead of $(\overline{G2})$, provided that we add the assumption that $\|\tilde{h}\|_{L^1(0,T)}$ is sufficiently small. Also, in case of Theorem 3.3, the truncation must be done in such a way that (ii) still holds for truncated functions.

ii) We have studied the case where the behaviour of $g(u)$ for u negative and large has the main role. Of course, results of the same type hold if we consider a nonlinearity whose behaviour at $\pm\infty$ is reversed with respect to the one just considered.

iii) Theorems 3.2 and 3.3 may be related to Theorem 1 of Fernandes and Zanolin [7] and Theorem 2 of Fonda [8]. These are sharp results where the asymptotic behaviour of g is described simply by $\liminf\limits_{u \to -\infty} \dfrac{2\hat{G}u}{u^2} < \dfrac{\pi^2}{T^2}$. Note, however, that our set of assumptions are different: namely, we use

$(\overline{G2})$, and we do not use coerciveness of $\hat{G}$. Note also that, for some functions $\hat{g}$ the $\liminf$ in the left-hand side of (3.13) may be smaller than $\liminf_{u \to -\infty} \dfrac{2\hat{G}(u)}{u^2}$.

3.2 The discontinuous case

We are now in a position to study problem (1.1) for functions with a jump discontinuity at $u = 0$, that is, we consider the following assumption:

(D2) $g : \mathbb{R} \to \mathbb{R}$ is a continuous function in $\mathbb{R} \setminus \{0\}$ and the limits

$$g(0^{\pm}) = \lim_{x \to 0^{\pm}} g(x)$$

exist and are finite.

Theorem 3.4 *Let g satisfy $(\overline{G2})$, (D2) and*

$$\limsup_{u \to -\infty} \frac{g(u)}{u} < \frac{\pi^2}{T^2}.$$

Assume in addition that either

(i) g is bounded below in $[0, +\infty)$, or

(ii) $G(u) := \displaystyle\int_0^u g(s)\,ds - \bar{h}u$ is bounded below in $[0, +\infty)$.

Then the problem (1.1) has a generalized solution.

Proof. We construct a sequence of continuous functions $\{g_n\}_{n=1}^{\infty}$ verifying (2.6) and linear in $[-\frac{1}{n}, \frac{1}{n}]$. Clearly, these are admissible with respect to one of them, say, g_1. For this function the assumptions of Theorem 3.2 hold. Hence the approximated problems

$$u'' + g_n(u) = h(t), \quad u(0) = u(T),\ u'(0) = u'(T),$$

have solutions u_n and moreover there are constants c, C such that $c \le u_n(t) \le C$ for all $t \in [0, T]$ and for all $n \in \mathbb{N}$. Using Ascoli's theorem in a standard manner we extract a convergent subsequence whose limit can be shown, as in the proof of Theorem 2.6, to be a generalized solution of (1.1). $\qquad\square$

In the same way we obtain the following result.

Theorem 3.5 *Let g satisfy $(\overline{G2})$, (D2) and*

$$\liminf_{u \to -\infty} \frac{\min_{u \le s \le 0} g(s)}{u} < \frac{2\pi}{T^2}.$$

Assume in addition that either

(i) g is bounded below in $[0, +\infty)$, or

(ii) $G(u) := \displaystyle\int_0^u g(s)\,ds - \bar{h}u$ is bounded below in $[0, +\infty)$.

Then the problem (1.1) has a generalized solution.

Remark 3.2 *We can also replace (G2)' for $(\overline{G2})$ to obtain solutions for $\|\tilde{h}\|_{L^1(0,T)}$ small, cf. Remark 3.1.*

Acknowledgement

References

[1] Ambrosetti, A. and Badiale, M., The dual variational principle and elliptic problems with discontinuous nonlinearities, *J. Math. Anal. Appl.*, 140, 363-373, (1989).

[2] Avgerinos, E. P., Papageorgiou, N. S. and Yannakakis, N., Periodic solutions for second order differential inclusions with nonconvex and unbounded multifunction, *Acta Math. Hungar.*, 83, no. 4, 303-314, (1999).

[3] Bertone, A. M. and Gonçalves, J. V., Discontinuous elliptic problems in $\mathbf{R}^N$: lower and upper solutions and variational principles, *Discrete Contin. Dynam. Systems*, 6, no. 2, 315-328, (2000).

[4] Cabada, A. and Sanchez, L., Second order singular periodic problems in presence of dry friction, *Proc. Amer. Math. Soc.*, 131, no. 7, 2137-2144, (2003).

[5] Cingolani, S. and Lazzo, M. Discontinuous elliptic problems in $\mathbb{R}^N$ without monotonicity assumptions, *Comment. Math. Univ. Carolin.*, 42, no. 3, 451-458, (2001).

[6] De Coster, C. and Tarallo, M., Foliations, associated reductions and lower and upper solutions, *Calc. Var.*, 15, 25-44, (2002).

[7] Fernandes, L. and Zanolin, F., Periodic solutions of a second order differential equation with one-sided growth restrictions on the restoring term, *Arch. Math.*, 51, 151-163, (1988).

[8] Fonda, A., On the existence of periodic solutions for scalar second order differential equations when only the asymptotic behaviour of the potential is known, *Proc. Amer. Math. Soc.*, 119, no. 2, 439-445, (1993).

[9] Hu, S., Kandilakis, D. A. and Papageorgiou, N. S., Periodic solutions for nonconvex differential inclusions, *Proc. Amer. Math. Soc.*, 127, no. 1, 89-94, (1999).

[10] Iannacci, R. and Nkashama, M. N., Unbounded perturbations of forced second order ordinary differential equations at resonance, *J. Differential Equations*, 69, no. 3, 289-309, (1987).

[11] Li, Y. and Lin, Z. H., Periodic solutions of differential inclusions, *Nonlinear Anal.*, 24, no. 5, 631-641, (1995).

[12] Li, G. and Xue, X., On the existence of periodic solutions for differential inclusions, *J. Math. Anal. Appl.*, 276, no. 1, 168-183, (2002).

[13] Mawhin, J., Problemes de Dirichlet variationnels non lineaires, Les Presses de l'Universite de Montreal, (1987).

[14] Mawhin, J. and Ward, J. R., Jr. Periodic solutions of some forced Liénard differential equations at resonance, *Arch. Math.*, 41, no. 4, 337-351, (1983).

[15] Pasca D., Periodic solutions for second order differential inclusions, *Comm. Appl. Nonlinear Anal.*, 6, no. 4, 91-98, (1999).

[16] Reissig, R., Periodic solutions of a second-order differential equation including a one-sided restoring term, *Arch. Math.*, 33, no. 1, 85-90, (1979).

[17] Schmitt, K., Periodic solutions of a forced nonlinear oscillator involving a onesided restoring force, *Arch. Math.*, 31, no. 1, 70-73, (1978).

[18] Ward, J. R., Jr., Periodic solutions of ordinary differential equations with bounded nonlinearities, *Topol. Methods Nonlinear Anal.*, 19, 275-282, (2002).