A generalization of Montel–Tonelli's Uniqueness Theorem

José Ángel Cid[†] and Rodrigo López Pouso[‡*]

† Departamento de Matemáticas, Universidade de Vigo,

32004, Pabellón 3, Campus de Ourense, Spain.

Email: angelcid@uvigo.es

‡ Departamento de Análise Matemática, Universidade de Santiago de Compostela,

15782, Facultade de Matemáticas, Campus Vida, Santiago, Spain.

Email: rodrigo.lopez@usc.es

Abstract

We present a new uniqueness result for first order systems of ordinary differential equations which contains a generalization of Montel–Tonelli's Uniqueness Theorem as a particular case. An example is given to illustrate its applicability.

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1 Introduction

Let $a, b \in (0, +\infty)$, $U = [t_0, t_0 + a] \times \{x \in \mathbb{R}^n : \|x - x_0\| \le b\}$ and let $f : U \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be continuous on U. This paper considers uniqueness of solutions for the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0. \tag{1.1}$$

One of the more general uniqueness theorems is due to Kamke, see [1, Theorem 3.8.1] or [5, Theorem 6.1], who improved an earlier version by Perron.

Theorem 1.1 *[Kamke's Uniqueness Theorem] Assume that for all (t, x), $(t, y) \in U$, $t \neq t_0$, we have*

$$\|f(t, x) - f(t, y)\| \le g(t - t_0, \|x - y\|), \tag{1.2}$$

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for some continuous function $g : (0,a] \times [0,2b] \longrightarrow [0,+\infty)$ *such that for every* $c \in (0,a)$, $x(t) \equiv 0$ *is the only solution of*

$$x'(t) = g(t,x(t)) \quad \text{for all } t \in (0,c), \text{ and } \lim_{t\to 0^+} \frac{x(t)}{t} = 0. \tag{1.3}$$

Then the initial value problem (1.1) has at most one solution in $[t_0, t_0 + a]$.

While many uniqueness results follow from Kamke's Theorem (e.g. Lipschitz's, Osgood's, and many more, see [1, Corollary 1.15.6]), its conditions are in general difficult to check. In this paper we introduce novel conditions on the function $g(t,x)$ which imply that it satisfies the assumptions in Kamke's Theorem, thus getting new applicable criteria for the uniqueness of (1.1). In particular, we are going to improve on Montel–Tonelli's Uniqueness Theorem [1, Theorem 1.5.1].

Theorem 1.2 *[Montel–Tonelli's Uniqueness Theorem] Assume that for all* (t,x), $(t,y) \in U$, $t \neq t_0$, *we have*

$$\|f(t,x) - f(t,y)\| \le p(t-t_0)\psi(\|x-y\|), \tag{1.4}$$

where $p : (0,a] \longrightarrow (0,+\infty)$ *is integrable, and* $\psi : [0,2b] \longrightarrow [0,+\infty)$ *is continuous and*

$$\int_{0^+} \frac{dr}{\psi(r)} = +\infty. \tag{1.5}$$

Then the initial value problem (1.1) has at most one solution in $[t_0, t_0 + a]$.

2 New uniqueness criteria

The main result in this paper extends Montel–Tonelli's Uniqueness Theorem by letting ψ depend also on the t argument in (1.4).

Theorem 2.1 *A continuous function* $g : (0,a] \times [0,2b] \longrightarrow [0,+\infty)$ *satisfies the conditions of Kamke's Uniqueness Theorem provided that* $g(t,x) > 0$ *if* $x > 0$ *and* $g(t,x) \le p(t)\psi(t,x)$, *where* $p : (0,a] \longrightarrow (0,+\infty)$ *is measurable,* $\psi : (0,a] \times [0,2b] \to [0,+\infty)$ *is continuous and satisfies the following properties:*

(i) $\psi(t,0) = 0$ *for all* $t \in (0,a]$;

(ii) ψ *is nonincreasing with respect to its first variable; and*

(iii) For every increasing and differentiable function $u : (t_1,t_2] \subset (0,a] \longrightarrow (0,2b)$ *such that*

$$\lim_{t\to t_1^+} \frac{u(t)}{t-t_1} = 0, \tag{2.6}$$

we have

$$\limsup_{t\to t_1^+} \left(\int_{u(t)}^{u(t_2)} \frac{dr}{\psi(r,r)} - \int_t^{t_2} p(s)\,ds \right) > 0. \tag{2.7}$$

Proof. Reasoning by contradiction, we assume that $x(t)$ is a nontrivial solution of (1.3). We can then find $t_1, t_2 \in [0,a]$ such that $x(t_1) = 0$ and $x(t) > 0$ for all $t \in (t_1, t_2]$.

The assumptions imply that $x'(t) = g(t,x(t)) > 0$ for all $t \in (t_1, t_2]$, and therefore x is increasing.

If $t_1 = 0$, then $x(t)$ satisfies (2.6) because it is a solution of (1.3); otherwise, we deduce from L'Hôpital's rule that

$$\lim_{t \to t_1^+} \frac{x(t)}{t - t_1} = \lim_{t \to t_1^+} x'(t) = \lim_{t \to t_1^+} g(t, x(t)) = g(t_1, 0) = 0.$$

Hence $x(t)$ satisfies (2.6). In particular, we can assume without loss of generality that $0 < x(t) < t$ for all $t \in (t_1, t_2]$ and, therefore, $x^{-1}(r) > r$ for all $r \in (0, x(t_2)]$.

For every $t \in (t_1, t_2)$ we have $x'(t) \le p(t)\psi(t, x(t))$, and then the change of variables formula yields

$$\int_t^{t_2} p(s)ds \ge \int_t^{t_2} \frac{x'(s)}{\psi(s, x(s))} ds = \int_{x(t)}^{x(t_2)} \frac{dr}{\psi(x^{-1}(r), r)}.$$

Since ψ is nonincreasing with respect to its first argument, and $x^{-1}(r) > r$ on $(0, x(t_2)]$, we deduce that

$$\int_t^{t_2} p(s)ds \ge \int_{x(t)}^{x(t_2)} \frac{dr}{\psi(r, r)} \quad \text{for every } t \in (t_1, t_2],$$

and we obtain a contradiction with (2.7) with $u(t) = x(t)$. □

Theorem 2.1 is a variant of [2, Theorem 3.1], less general but easier to prove and, we intend to show, easier to apply. Taking $p(t) = 1/t$ and $\psi(t, x) = x$ in Theorem 2.1 we obtain as a particular case the celebrated Nagumo's uniqueness theorem, see [1, Theorem 1.6.2], that has attracted a renewed interest in recent years, see [3, 4, 6, 7]. As an immediate consequence of Theorem 2.1 we also obtain the following generalization of Montel–Tonelli's Uniqueness Theorem.

Theorem 2.2 *Problem (1.1) has at most one solution in* $[t_0, t_0 + a]$ *provided that*

$$\|f(t, x) - f(t, y)\| \le p(t - t_0)\psi(t - t_0, \|x - y\|) \quad \textit{for all } (t, x), (t, y) \in U,\ t \ne t_0, \tag{2.8}$$

where $p : (0, a] \longrightarrow (0, +\infty)$ *is integrable,* $\psi : (0, a] \times [0, 2b] \longrightarrow [0, +\infty)$ *is continuous,* ψ *satisfies conditions* (i) *and* (ii) *in Theorem 2.1, and*

$$\int_{0+} \frac{dr}{\psi(r, r)} = +\infty. \tag{2.9}$$

Theorem 2.2 is a strict generalization of Montel–Tonelli's Theorem as we emphasize in our next corollary. Notice that it is based on conditions which replace Lipschitz constants by certain nonintegrable functions of t, thus falling outside the scope of Montel–Tonelli's Theorem, and providing us with a criterion in the spirit of Nagumo's.

Corollary 2.1 *Problem (1.1) has at most one solution in* $[t_0, t_0 + a]$ *provided that for some* $c > 0$ *we have*

$$\|f(t, x) - f(t, y)\| \le \frac{c}{(t - t_0)h(t - t_0)} \|x - y\| \quad \textit{for all } (t, x), (t, y) \in U,\ t \ne t_0, \tag{2.10}$$

where $h : (0, a] \to (0, +\infty)$ *is continuous, nonincreasing and satisfies*

$$\int_{0+} \frac{1}{t\, h^2(t)} dt < +\infty \quad \textit{and} \quad \int_{0+} \frac{1}{t\, h(t)} dt = +\infty.$$

Proof. Use Theorem 2.2 with $p(t) = \dfrac{c}{th^2(t)}$ and $\psi(t, x) = h(t)\, x$. □

The particular choice $h(t) = 1 - \ln t$ for $t \in (0, e)$ provides the following consequence.

Corollary 2.2 *Problem (1.1) has at most one solution in* $[t_0, t_0 + a]$ *for any* $a \in (0, e)$*, provided that for some* $c > 0$ *we have*

$$\|f(t,x) - f(t,y)\| \leq \frac{c}{(t-t_0)(1-\ln(t-t_0))}\|x-y\| \quad \textit{for all } (t,x),(t,y) \in U,\ t \neq t_0. \tag{2.11}$$

Other functions can be used in (2.11) instead of $c/[(t-t_0)(1-\ln(t-t_0))]$.

Corollary 2.3 *Problem (1.1) has at most one solution in some* $[t_0, t_0 + \hat{a}] \subset [t_0, t_0 + a]$ *provided that*

$$\|f(t,x) - f(t,y)\| \leq q(t-t_0)\|x-y\| \quad \textit{for all } (t,x),(t,y) \in U,\ t \neq t_0, \tag{2.12}$$

where $q : (0, a] \longrightarrow (0, +\infty)$ *is continuous and*

$$\limsup_{t \to 0^+} q(t)\, t\,(1 - \ln t) < +\infty. \tag{2.13}$$

Proof. Condition (2.13) implies the existence of constants $c > 0$ and $\hat{a} \in (0, a]$ such that

$$q(t) \leq \frac{c}{t\,(1-\ln t)} \quad \text{for all } t \in (0, \hat{a}],$$

and therefore we can apply Theorem 2.2 on $[t_0, t_0 + \hat{a}]$. □

Finally, we show an example of the applicability of Corollaries 2.2 and 2.3.

Example 2.1 *We are going to study the existence and uniqueness of solutions for*

$$x' = \sqrt{\alpha\,|x| + \beta^2\,t^2\,(\gamma - \delta \ln t)^2}, \quad t > 0, \quad x(0) = 0, \tag{2.14}$$

where α, β, γ *and* δ *are positive constants. Notice that we have more than one solution if* $\beta = 0$*, e.g.* $x(t) = 0$ *and* $x(t) = \alpha t^2/4$.

First, we rewrite the previous problem in terms of (1.1) with $t_0 = 0$*: we fix* $a \in (0, 1)$ *and* $b > 0$*, and we define*

$$f(t,x) = \begin{cases} \sqrt{\alpha\,|x| + \beta^2\,t^2\,(\gamma - \delta \ln t)^2}, & \textit{if } (t,x) \in (0,a] \times [-b,b], \\ \sqrt{\alpha\,|x|}, & \textit{if } t = 0 \textit{ and } x \in [-b,b]. \end{cases}$$

This function f *is continuous on* $[0, a] \times [-b, b]$*. Since* $f(t, x)$ *is sublinear in* x*, we can assume that* $b > 0$ *is sufficiently large so that (1.1) with* $t_0 = 0$ *has at least one solution on the whole of* $[0, a]$*, and such a solution is necessarily positive on* $(0, a]$*.*

Now for the uniqueness. If $t \in (0, a]$ *and* $|x| < |y| \leq b$*, then there is some* $z \in (|x|, |y|)$ *such that*

$$|f(t,x) - f(t,y)| = \frac{\alpha}{2\sqrt{\alpha\,z + \beta^2\,t^2\,(\gamma - \delta \ln t)^2}}\big||x| - |y|\big| \leq \frac{\alpha}{2\beta\,t\,(\gamma - \delta \ln t)}|x - y|.$$

Therefore, we can use Corollary 2.3 with

$$q(t) = \frac{\alpha}{2\beta\,t\,(\gamma - \delta \ln t)},$$

to ensure that (1.1) has a unique solution on some $[0, \hat{a}] \subset [0, a]$ *(in fact we have uniqueness on* $[0, a]$*: the solution* $x(t)$ *cannot bifurcate at any* $t \in (\hat{a}, a]$ *because the classical Lipschitz's Theorem applies in neighborhoods of points* $(t, x(t)) \in (0, a) \times (-b, b)$ *if* $x(t) > 0$*).*

The case $\gamma = \delta = 1$ *is easier because for all* $t \in (0, a]$ *and all* $x, y \in [-b, b]$ *we have*

$$|f(t,x) - f(t,y)| \le \frac{\alpha}{2\beta\, t\,(1 - \ln t)} |x - y|, \tag{2.15}$$

and therefore (1.1) has at most one solution on $[0, a]$ *by virtue of Corollary 2.2.*

Notice that (2.15) implies that for a sufficiently small $\hat{a} \in (0, a]$ *we have*

$$|f(t,x) - f(t,y)| \le \frac{|x-y|}{t} \quad \text{for all } t \in (0, \hat{a}] \text{ and } x, y \in [-b, b], \tag{2.16}$$

which implies uniqueness on $[0, \hat{a}]$ *via Nagumo's Uniqueness Theorem, see [1, Theorem 1.6.2]. However, it is important to note that if* $\alpha/(2\beta(1 - \ln a)) > 1$ *then we cannot use Nagumo's Theorem to ensure uniqueness on the whole interval* $[0, a]$ *at one stroke, as we did before with Corollary 2.2.*

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