

SOLVABILITY OF SOME Φ -LAPLACIAN SINGULAR DIFFERENCE EQUATIONS DEFINED ON THE INTEGERS

ALBERTO CABADA AND JOSÉ ÁNGEL CID

ABSTRACT. This paper is devoted to proving the existence of at least one solution of the following boundary value problem

$$\begin{cases} \Delta(\Phi(\Delta u(k))) = f(k+1, u(k), u(k+1), u(k+2)) & \text{on } \mathbb{Z}, \\ u(-\infty) = -1, \quad u(+\infty) = 1, \end{cases}$$

where the function Φ is a homeomorphism from the finite interval $(-a, a)$ onto \mathbb{R} .

1. INTRODUCTION AND PRELIMINARY RESULTS

The study of difference equations represents a very important field in mathematical research. Different mathematical models coupled with the basic theory of this type of equation can be found in the classical monograph by S. Goldberg [18] and in the more recent books by V. Lakshmikantham and D. Trigiante [19] and S. Elaydi [16]. The study of the existence of solutions for first and second order difference equations coupled with different kinds of boundary value conditions can be found in [1, 2, 3, 4, 8, 11, 13, 22], among others.

In [10] we have studied the existence of heteroclinic connections for the differential equation

$$\begin{cases} (\Phi(u'(t)))' = f(t, u(t), u'(t)), & \text{a.e. on } \mathbb{R}, \\ u(-\infty) = -1, \quad u(+\infty) = 1. \end{cases} \quad (1)$$

where $\Phi : (-a, a) \rightarrow \mathbb{R}$ is an increasing homeomorphism with $\Phi(0) = 0$. In this case, the operator Φ is called *singular* in the terminology introduced by Bereanu and Mawhin [5] and its model is the relativistic operator $\Phi(s) = \frac{s}{\sqrt{1-s^2}}$ for $s \in (-1, 1)$. Problem (1) with an increasing homeomorphism $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ has been studied in [7]. In this case, when Φ is the identity, the problem is motivated by the search for traveling wave solutions for reaction-diffusion equations [1, 21]. On the other hand, when $\Phi(s) = |s|^{p-2}s, p > 1$, we obtain the one-dimensional p-laplacian to which many papers have been devoted (see for instance [14, 17, 23] and references therein). Recently, this study has also been developed for difference equations in [6, 9, 12].

In this paper we deal with the discrete version of the problem (1), that is

$$\Delta(\Phi(\Delta u(k))) = f(k+1, u(k), u(k+1), u(k+2)), \quad \text{on } \mathbb{Z}, \quad (2)$$

Received January 11, 2009; Accepted January 27, 2009.
 2000 *Mathematics Subject Classification.* 39A10, 39A11, 39A70.

Key words and phrases. Singular Φ -Laplacian, discrete boundary value problem, solutions defined on unbounded intervals.

This work was partially supported by Ministerio de Educación y Ciencia, Spain, project MTM2007-61724, and by Xunta de Galicia, Spain, project PGIDIT06PXIB207023PR.

$$u(-\infty) = -1, \quad u(+\infty) = 1. \quad (3)$$

In order to prove the existence of a solution we firstly deduce an analogous result to [5, Corollary 1] for discrete problems, namely, the existence of solution for the singular Dirichlet discrete problem in a finite interval for each arbitrary continuous function. By using such a property, we construct a suitable sequence of functions that converges to a solution of our problem.

In the sequel we assume the following:

(h0) $\Phi : (-a, a) \rightarrow \mathbb{R}$ is an increasing homeomorphism with $0 < a < +\infty$ (i.e., Φ is singular).

(f0) $f : \mathbb{Z} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous and satisfies the symmetry condition

$$f(k, x, y, z) = -f(-k, -z, -y, -x) \quad \text{for all } k \in \mathbb{Z} \text{ and } (x, y, z) \in \mathbb{R}^3.$$

(f1) $f(k, x, 1, z) = 0 = f(k, x, -1, z)$ for all $k \in \mathbb{Z}$ and $(x, z) \in \mathbb{R}^2$.

(f2) $f(k, x, y, z) < 0$ for all $k \in \{1, 2, \dots\}$ and $\max\{|x|, |y|, |z|\} < 1$. (Moreover for every compact set $K \subset (0, 1)^3$, there exists $\kappa \in \mathbb{N}$ and $h_K : \{\kappa, \kappa + 1, \dots\} \rightarrow \mathbb{R}$ such that

$$f(k, x, y, z) \leq h_K(k) \quad \text{for all } k \geq \kappa \text{ and } (x, y, z) \in K,$$

and

$$\sum_{s=\kappa}^{+\infty} h_K(s) = -\infty.$$

A solution of (2) – (3) is a function $u : \mathbb{Z} \rightarrow \mathbb{R}$ such that $\Delta u(k) \in (-a, a)$ for all $k \in \mathbb{Z}$ and u satisfies the difference equation (2) coupled with the boundary conditions (3).

We shall approximate problem (2)–(3) by problems defined on finite sets. We first prove the uniqueness of solution of the following equation:

$$\Delta(\Phi(\Delta u(k))) = g(k), \quad \text{for all } k \in J_n \equiv \{0, \dots, n-2\}, \quad u(0) = u(n) = 0. \quad (4)$$

Lemma 1.1. *Suppose that $\Phi : (-a, a) \rightarrow \mathbb{R}$ satisfies (h0). Then, for all $g : J_n \rightarrow \mathbb{R}$ the Dirichlet problem (4) has a unique solution. (Notice that in particular $\|\Delta u\|_\infty < a$).*

Proof. By direct computation, one can verify that every solution of Problem (4) satisfies the following equality

$$u_g(k) = \sum_{j=0}^{k-1} \Phi^{-1} \left(\tau_g + \sum_{s=0}^{j-1} g(s) \right), \quad \text{for all } k \in I_n \equiv \{0, \dots, n\}, \quad (5)$$

τ_g being a solution of the following expression

$$F_g(\tau) := \sum_{j=0}^{n-1} \Phi^{-1} \left(\tau + \sum_{s=0}^{j-1} g(s) \right) = 0. \quad (6)$$

The fact that the function $F_g : \mathbb{R} \rightarrow \mathbb{R}$ has a unique real root is deduced from the following facts:

- (1) The continuity of Φ^{-1} implies the continuity of function F_g .
- (2) Since Φ^{-1} is strictly increasing the same holds for function F_g .
- (3) $\lim_{\tau \rightarrow +\infty} F_g(\tau) = na > 0$.
- (4) $\lim_{\tau \rightarrow -\infty} F_g(\tau) = -na < 0$.

It is obvious that the uniqueness of constant τ_g is equivalent to the uniqueness of the solution u_g . \square

Now we are in a position to prove the analogous result for discrete equations proved by Bereanu and Mawhin in [5, Corollary 1]. The result is the following:

Theorem 1.2. *Suppose that $\Phi : (-a, a) \rightarrow \mathbb{R}$ satisfies (h0). Then, for all continuous functions $f : J_n \times \mathbb{R}^3 \rightarrow \mathbb{R}$ the Dirichlet problem*

$$\Delta(\Phi(\Delta u(k))) = f(k+1, u(k), u(k+1), u(k+2)), \quad k \in J_n, \quad u(0) = u(n) = 0, \quad (7)$$

has at least one solution. Moreover, any solution of problem (7) satisfies that $\|\Delta u\|_\infty < a$.

Proof. Let $T : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ defined as

$$T u(k) = \sum_{j=0}^{k-1} \Phi^{-1} \left(\tau_u + \sum_{s=0}^{j-1} f(s+1, u(s), u(s+1), u(s+2)) \right), \quad k \in I_n, \quad (8)$$

with τ_u the unique solution of the following expression

$$g_u(\tau) := \sum_{j=0}^{n-1} \Phi^{-1} \left(\tau + \sum_{s=0}^{j-1} f(s+1, u(s), u(s+1), u(s+2)) \right) = 0. \quad (9)$$

From Lemma 1.1 we know that operator T is well defined and that the fixed points of operator T are the solutions of problem (7).

First we prove that operator T is continuous: suppose $u_m \rightarrow u$ in \mathbb{R}^{n+1} and let τ_m be the corresponding value for u_m given by (9) and τ_u associated to u . Let us see that $\lim_{m \rightarrow \infty} \tau_m = \tau_u$.

By construction of τ_m and τ_u we have that for all $m \in \mathbb{N}$:

$$\begin{aligned} 0 &= \sum_{j=0}^{n-1} \Phi^{-1} \left(\tau_u + \sum_{s=0}^{j-1} f(s+1, u(s), u(s+1), u(s+2)) \right) \\ &= \sum_{j=0}^{n-1} \Phi^{-1} \left(\tau_m + \sum_{s=0}^{j-1} f(s+1, u_m(s), u_m(s+1), u_m(s+2)) \right). \end{aligned} \quad (10)$$

Since $\{u_m\}$ is a convergent sequence to u we have that

$$\{(k+1, u_m(k), u_m(k+1), u_m(k+2)), m \in \mathbb{N}\} \cup \{(k+1, u(k), u(k+1), u(k+2))\}$$

is a compact set in $J_n \times \mathbb{R}^3$. As consequence, $\{f(k+1, u_m(k), u_m(k+1), u_m(k+2))\}_{m \in \mathbb{N}}$ is a bounded sequence in \mathbb{R}^{n+1} and, from (10) the sequence $\{\tau_m\}$ is bounded too, and then we conclude that there exists a subsequence $\{\tau_{m_k}\}$ converging to a real number $\gamma = \limsup \{\tau_m\}$.

Thus, from the continuity of Φ^{-1} and f , we arrive at

$$\begin{aligned} \sum_{j=0}^{n-1} \Phi^{-1} \left(\tau_u + \sum_{s=0}^{j-1} f(s+1, u(s), u(s+1), u(s+2)) \right) &= \\ \sum_{j=0}^{n-1} \Phi^{-1} \left(\gamma + \sum_{s=0}^{j-1} f(s+1, u(s), u(s+1), u(s+2)) \right), \end{aligned}$$

and since Φ^{-1} is a strictly increasing function, we conclude that $\tau_u = \gamma$.

Analogously, we verify that $\tau_u = \liminf \{\tau_m\}$.

Now, by the continuity of f it follows that

$$\lim_{m \rightarrow \infty} T u_m(k) = T u(k) \quad \text{for all } k \in I_n,$$

which is equivalent to say that T is a continuous operator in \mathbb{R}^{n+1} .

Finally, it is clear that $\|Tu\|_\infty \leq na$ for each $u \in \mathbb{R}^{n+1}$ and then Brouwer fixed point theorem (see [20]) ensures the existence of at least one fixed point of the operator T and, as a consequence, the existence of at least one solution of problem (7).

Notice that, by using expression (8), we have that for all $k \in I_n$ that

$$\Delta u(k) = \Phi^{-1} \left(\tau_u + \sum_{s=0}^{j-1} f(s+1, u(s), u(s+1), u(s+2)) \right) \in (-a, a).$$

□

2. MAIN RESULT

Next we prove the solvability of the singular Φ -laplacian problem (2)–(3).

Theorem 2.1. *If conditions (h0), (f0), (f1) and (f2) hold then problem (2)–(3) has an odd increasing solution $u : \mathbb{Z} \rightarrow \mathbb{R}$.*

Proof. By the symmetry condition (f0) it is enough to prove the existence of a solution $u : \mathbb{N} \rightarrow \mathbb{R}$ of (2) satisfying $u(0) = 0$ and $\lim_{k \rightarrow \infty} u(k) = 1$, since its odd extension solves (2)–(3).

Claim 1. *For each $n \in \mathbb{N}$ the Dirichlet boundary value problem*

$$\Delta(\Phi(\Delta u(k))) = f(k+1, u(k), u(k+1), u(k+2)), k \in J_n, \quad u(0) = u(n) = 0, \quad (11)$$

has a solution $u_n : I_n \rightarrow \mathbb{R}$ satisfying $0 \leq u_n(k) \leq 1$ and $\|\Delta u_n\|_\infty < a$.

Let $\tilde{f} : \mathbb{N} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined as

$$\tilde{f}(k, x, y, z) = \begin{cases} f(k, x, y, z), & \text{if } -1 \leq y \leq 1 \\ 0, & \text{in other case.} \end{cases}$$

It is clear, from condition (f1), that \tilde{f} is a continuous function. On the other hand, for each $n \in \mathbb{N}$ the modified problem

$$\Delta(\Phi(\Delta u(k))) = \tilde{f}(k+1, u(k), u(k+1), u(k+2)), u(0) = 0 = u(n),$$

has by Theorem 1.2 a solution $u_n : I_n \rightarrow \mathbb{R}$ with $\|\Delta u_n\|_\infty < a$.

If there is $k_0 \in \{1, \dots, n-1\}$ such that $u(k_0) > 1$ then, from the definition of \tilde{f} , we arrive at

$$\Delta u(k_0 - 1) = \Delta u(k_0).$$

So, by recurrence, we deduce that if $\Delta u(k_0) \geq 0$ then $u(k) \geq u(k_0) > 1$ for all $k \in \{k_0, \dots, n\}$. In case $\Delta u(k_0) < 0$ holds we deduce that $u(k) \geq u(k_0) > 1$ for all $k \in \{0, \dots, k_0\}$. In both cases we attain a contradiction with $u_n(0) = 0 = u_n(n)$.

Analogously, we deduce that $u_n(k) \geq -1$ for all $k \in I_n$ and then u_n is a solution of (11).

On the other hand, if $\Delta u(0) < 0$ condition (f2) implies that $u_n(k) < 0$ for all $k \in I_n$, which is again a contradiction. Therefore $0 \leq u_n(k) \leq 1$.

Claim 2. *There exists a bounded nondecreasing solution $u : \mathbb{N} \rightarrow \mathbb{R}$ of (2) such that $u(0) = 0$ and $0 \leq u(k) \leq 1$.*

Since u_n is uniformly bounded, it is easy to prove that a subsequence of u_n converges uniformly on finite subsets to a solution $u : \mathbb{N} \rightarrow \mathbb{R}$ of (2).

Clearly $u(0) = 0$ and $0 \leq u(k) \leq 1$ for all $k \in \mathbb{N}$. As consequence of condition (f2) we have that $\Delta(\Phi(\Delta u(k))) \leq 0$, i. e., Δu is a nonincreasing function.

If there is $k_0 \in \mathbb{N}$ such that $\Delta u(k_0) < 0$ then we deduce that $\Delta u(k) \leq \Delta u(k_0) < 0$ for all $k \geq k_0$ and consequently $\lim_{k \rightarrow \infty} u(k) = -\infty$, a contradiction. Thus $\Delta u(k) \geq 0$ for all $k \geq 0$ and then u is nondecreasing in \mathbb{N} .

Claim 3. $\lim_{k \rightarrow +\infty} \Delta u(k) = 0$.

Since Δu is decreasing there exists $\lim_{k \rightarrow +\infty} \Delta u(k) \in \mathbb{R} \cup \{-\infty\}$. But as u is bounded, we deduce that $\lim_{k \rightarrow +\infty} \Delta u(k) = 0$.

Claim 4. $\lim_{k \rightarrow +\infty} u(k) = 1$.

Since u is nondecreasing and bounded there exists $\lim_{k \rightarrow +\infty} u(k) = l \in (0, 1]$. Suppose that $l < 1$. From (f2) and the facts that $0 \leq u(k) \leq l < 1$ and $\lim_{k \rightarrow +\infty} \Delta u(k) = 0$, it follows that there exists a related $\kappa > 0$ for which $0 < \Delta u(\kappa) < 1$ and a h_K (for some suitable compact set $K \subset (0, 1)^3$) such that

$$\Delta(\Phi(\Delta u(k))) = f(k+1, u(k), u(k+1), u(k+2)) \leq h_K(k+1) \quad \text{for all } k \geq \kappa,$$

and $\sum_{s=\kappa+1}^{\infty} h_K(s) = -\infty$.

But in this case

$$\lim_{k \rightarrow \infty} \Phi(\Delta u(k)) = -\infty$$

and then

$$\lim_{k \rightarrow \infty} \Delta u(k) = -a < 0,$$

which is a contradiction. Thus $l = 1$ and the proof finishes. □

Example 2.2. Let $m, n, p \in \mathbb{N}$ be given. Consider the problem

$$\begin{cases} \Delta \left(\frac{\Delta u(k)}{\sqrt{1-(\Delta u(k))^2}} \right) = k^{2l+1} (u(k))^{2m+1} (u(k)^{2p} - 1)^n (u(k+2))^{2m+1}, & \text{on } \mathbb{Z}, \\ u(-\infty) = -1, \quad u(+\infty) = 1, \end{cases}$$

where $\Phi(s) = \frac{s}{\sqrt{1-s^2}}$ for all $s \in (-1, 1)$ and $f(k, x, y, z) = k^{2l+1} x^{2m+1} (y^{2p} - 1)^n z^{2m+1}$.

Clearly conditions of Theorem 2.1 are fulfilled and so its solvability is guaranteed.

REFERENCES

- [1] R. Agarwal and D. O'Regan, Infinite interval problems for differential, difference and integral equations, Kluwer Academic, Dordrecht, 2001.
- [2] C. Bereanu and J. Mawhin, Periodic solutions of first order nonlinear difference equations, Rend. Sem. Mat. Univ. Politec. Torino 65 (2007), 17–33.
- [3] C. Bereanu and J. Mawhin, Existence and multiplicity results for periodic solutions of nonlinear difference equations, J. Differ. Equations Appl. 12 (2006), 677–695.
- [4] C. Bereanu and J. Mawhin, Existence and multiplicity results for nonlinear second order difference equations with Dirichlet boundary conditions, Math. Bohem. 131 (2006), 145–160.
- [5] C. Bereanu and J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular Φ -Laplacian, J. Differential Equations 243 (2007), 536–557.

- [6] C. Bereanu and H. Thompson, Periodic solutions of second order nonlinear difference equations with discrete Φ -Laplacian, *J. Math. Anal. Appl.* 330 (2007), 1002–1015.
- [7] B. Bianconi and F. Papalini, Non-autonomous boundary value problems on the real line, *Discrete Contin. Dynam. Systems* 15 (2006), 759–776.
- [8] A. Cabada, The method of lower and upper solutions for periodic and anti-periodic difference equations, *Electron. Trans. Numer. Anal.* 27 (2007), 13–25.
- [9] A. Cabada, Extremal solutions for the difference Φ -Laplacian problem with nonlinear functional boundary conditions, *Advances in difference equations, III, Comput. Math. Appl.* 42 (2001), 593–601.
- [10] A. Cabada and J. Cid, Heteroclinic solutions for non-autonomous boundary value problems with singular Φ -Laplacian operators, to appear in *Discrete Contin. Dyn. Syst., suppl., Proceedings of the 7th AIMS International Conference on Dynamical Systems, Differential Equations and Applications, 2009*.
- [11] A. Cabada and V. Otero-Espinar, Fixed sign solutions of second-order difference equations with Neumann boundary conditions, *Advances in difference equations, IV, Comput. Math. Appl.* 45 (2003), 1125–1136.
- [12] A. Cabada and V. Otero-Espinar, Existence and comparison results for difference Φ -Laplacian boundary value problems with lower and upper solutions in reverse order, *J. Math. Anal. Appl.* 267 (2002), 501–521.
- [13] A. Cabada, V. Otero-Espinar and R. Pouso, Existence and approximation of solutions for first-order discontinuous difference equations with nonlinear global conditions in the presence of lower and upper solutions, *Comput. Math. Appl.* 39 (2000), 21–33.
- [14] A. Cabada, P. Habets and R. Pouso, Optimal existence conditions for Φ -Laplacian equations with upper and lower solutions in the reversed order, *J. Differential Equations* 166 (2000), 385–401.
- [15] H. Dang and S. Oppenheimer, Existence and uniqueness results for some nonlinear boundary value problems, *J. Math. Anal. Appl.* 198 (1996), 35–48.
- [16] S. Elaydi, *An introduction to difference equations*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1996.
- [17] C. Fabry and R. Manásevich, Equations with a p -Laplacian and an asymmetric nonlinear term, *Discrete Contin. Dynam. Syst.* 7 (2001), 545–557.
- [18] S. Goldberg, *Introduction to difference equations*, New York, John Wiley & Sons, 1960.
- [19] V. Lakshmikantham and D. Trigiante, *Theory of difference equations. Numerical methods and applications*, Mathematics in Science and Engineering, 181. Academic Press, Inc., Boston, MA, 1988.
- [20] N. Lloyd, *Degree Theory*, Cambridge Univ. Press., 73, Cambridge, 1978.
- [21] C. Marcelli and F. Papalini, Heteroclinic connections for fully non-linear non-autonomous second-order differential equations, *J. Differential Equations* 241 (2007), 160–183.
- [22] F. M. Atici and G. Guseinov, Positive periodic solutions for nonlinear difference equations with periodic coefficients, *J. Math. Anal. Appl.*, 232 (1999), 166–182.
- [23] M. del Pino, P. Drábek and R. Manásevich, The Fredholm alternative at the first eigenvalue for the one dimensional p -Laplacian, *J. Differential Equations* 151 (1999), 386–419.

UNIVERSITY OF SANTIAGO DE COMPOSTELA, DEPARTMENT OF MATHEMATICAL ANALYSIS, 15782 SANTIAGO DE COMPOSTELA, GALICIA, SPAIN.

E-mail address: alberto.cabada@usc.es

UNIVERSITY OF JAÉN, DEPARTMENT OF MATHEMATICS, CAMPUS LAS LAGUNILLAS, 23071 JAÉN, SPAIN.

E-mail address: angelcid@ujaen.es

**SOLVABILITY OF SOME ϕ -LAPLACIAN SINGULAR DIFFERENCE
EQUATIONS DEFINED ON THE INTEGERS**

ALBERTO CABADA AND JOSÉ ÁNGEL CID

الخلاصة (Abstract). هذه الورقة مخصصة لإثبات وجود حل واحد على الأقل لمسألة القيمة الحدية التالية

$$\begin{cases} \Delta(\Phi(\Delta u(k))) = f(k+1, u(k), u(k+1), u(k+2)) \text{ on } \mathbb{Z}, \\ u(-\infty) = -1, u(+\infty) = 1, \end{cases}$$

حيث تمثل الدالة Φ تشاكلاً مستمراً بين الفترة المحدودة $(-a, a)$ و \mathbb{R} .