Extremal solutions of ϕ-Laplacian – diffusion scalar problems with nonlinear functional boundary conditions in a unified way

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Abstract

The aim of this paper is to ensure the existence of extremal solutions lying between a pair of lower and upper solutions for a second order differential equation that includes both the ϕ–laplacian and the diffusion equations. The results hold from a suitable change of variables. An example is given to expose the applicability of the obtained results.

1 Introduction

The ϕ–Laplacian equations have appeared in the literature on the basis of the p– Laplacian equations, that follows, for some $p > 1$, a formulation of the type

$$-\frac{d}{dt}(|u'(t)|^{p-2}u'(t)) = f(t, u(t), u'(t)), \quad t \in I = [a, b].$$

This kind of problems models some physical phenomena in non-Newtonian fluid mechanics, see [11, 13, 19] and references therein. It is obvious that when $p = 2$ we are in the classical second order differential equation. Follow the qualitative properties of the function $\phi_p(x) = |x|^{p-2}x$, the p – Laplacian equations can be studied under the more general formulation of

$$-\frac{d}{dt}(\phi(u'(t))) = f(t, u(t), u'(t)), \quad t \in I,$$

with $\phi : \mathbb{R} \to \mathbb{R}$ an increasing homeomorphism. These problems have been studied exhaustively, under different points of view, by several authors in recent years [7, 8, 9, 10, 19, 20, 21]. One of the most useful techniques to approach such

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problems is given by the method of lower and upper solutions. This method allows us to ensure the existence of at least one solution (in some case, extremal solutions) of the considered problem.

On the other hand, the one dimensional diffusion equation is given by

$$(k(u(t))u'(t))' = f(t, u(t), u'(t)), \quad t \in I.$$

The function k is known as the diffusion coefficient and it is assumed that it depends on the concentration of the fluid. Under suitable assumptions this equation can be rewritten as

$$(W \circ u)''(t) = f(t, u(t), u'(t)), \quad t \in I,$$

where $W(u) := \int_0^u k(s)ds$.

We point out that the positive solutions of the nonlinear initial value problem

$$(u^{m+1}(t))'' + (t-1)u'(t) = 0, \ t \in (0,1), \ m > 0, \ u(0) = 0, \ \lim_{t \to 0^+} (u^{m+1}(t))' = 0,$$

have been studied in the semiconductor's production [15], water's filtration [17] and the transmission of a drug [1, 18].

A generalization of this equation has been treated in [2, 3, 4, 5, 6, 16]. In those papers different expressions of the nonlinear part of the equation have been boarded. In this situation the method of lower and upper solutions has been rarely used [14].

In this paper we present an operator that includes both expressions under the same formulation. This fact allows us to give some existence results for both problems. Before to do this in section 4, we present in section 3, some existence results for the ϕ − Laplacian equation that are new for this situation. To end the paper we give, in section 5, an example that point out the given results.

2 Preliminaries results

In this section, we consider the boundary value problem

$$\begin{cases} -[\phi(u')]'(t) = f(t, u(t), u'(t)) \quad \text{a.a. } t \in I = [a, b], \\ \\ L_1(u(a), u(b), u'(a), u'(b), u) = L_2(u(a), u(b)) = 0, \end{cases} \tag{2.1}$$

where $a < b$ are a pair of real numbers.

The results exposed in this section are taken from [8]. In a first moment, we consider the following assumptions:

(H1) $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, that is: $f(t, \cdot, \cdot)$ is a continuous function on $\mathbb{R}^2$ for a. a. $t \in I$; $f(\cdot, x, y)$ is a measurable for all $(x, y) \in \mathbb{R}^2$; for every $M > 0$ there exists a real-valued function $\psi_M \in L^1(I)$ such that for every $(x, y) \in \mathbb{R}^2$ with $\|(x, y)\|_\infty \le M$.

$$|f(t, x, y)| \le \psi_M(t) \quad \text{for a. a. } t \in I.$$

(H2) $\phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function.

(H3) $L_1 \in \mathcal{C}(\mathbb{R}^4 \times \mathcal{C}(I), \mathbb{R})$ is nondecreasing in the third variable, nonincreasing in the fourth and nondecreasing in the fifth one, i.e., if $\xi, \eta \in \mathcal{C}(I)$ are such that $\xi(t) \le \eta(t)$ for all $t \in I$ then

$$L_1(x, y, z, w, \xi) \le L_1(x, y, z, w, \eta).$$

On the other hand, $L_2 : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, it is nonincreasing with respect to its first variable and moreover for each $x \in \mathbb{R}$ the function $L_2(x, \cdot)$ is injective in $\mathbb{R}$.

Now we introduce the main concepts that we will use along the paper.

Definition 2.1 *We say that* $\alpha \in \mathcal{S}_1 := \{u \in \mathcal{C}^1(I) : \phi(u') \in AC(I)\}$ *is a lower solution of problem (2.1) if satisfies*

$$\begin{cases} -[\phi(\alpha')]'(t) \le f(t, \alpha(t), \alpha'(t)) \quad a.a.\ t \in I, \\ \\ L_1(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b), \alpha) \ge L_2(\alpha(a), \alpha(b)) = 0. \end{cases}$$

Analogously we say that $\beta \in \mathcal{S}_1$ *is an upper solution of problem (2.1) if the above inequalities are reversed.*

We say that $x \in \mathcal{S}_1$ *is a solution of problem (2.1) if it is both a lower and an upper solution.*

Whenever $\alpha \le \beta$ *we say that a solution* x^* *of problem (2.1) is the maximal solution in the set*

$$[\alpha, \beta] := \{u \in \mathcal{C}(I) : \alpha(t) \le u(t) \le \beta(t) \quad \text{for all} \ \ t \in I\},$$

if $x^* \in [\alpha, \beta]$ *and* $x^* \ge x$ *for any other solution* $x \in [\alpha, \beta]$.

The minimal solution in $[\alpha, \beta]$ *is defined analogously by reversing the inequalities. When both the minimal and the maximal solutions in* $[\alpha, \beta]$ *exist, we call them the extremal solutions in* $[\alpha, \beta]$.

Now, to ensure the existence of solutions of problem (2.1), we assume the following hypotheses.

(H4) There exist α, $\beta \in \mathcal{S}_1$ a lower and an upper solutions of problem (2.1) respectively, such that $\alpha \le \beta$ in I.

(H5) $f : I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies a Nagumo condition relative to the pair α and β, i.e.: there exist functions $h \in L^p(I)$, $1 \le p \le \infty$, and $\theta : [0, \infty) \to [0, \infty)$ continuous, satisfying

$$|f(t, u, v)| \le h(t)\theta(|v|) \quad \text{for a.a. } t \in I \text{ and all } u \in [\alpha(t), \beta(t)] \text{ and } v \in \mathbb{R},$$

$$\min\left\{ \int_{\phi(\nu)}^{\phi(\infty)} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du, \int_{\phi(-\infty)}^{\phi(-\nu)} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du \right\} > \mu^{\frac{p-1}{p}} \|h\|_p, \qquad (2.2)$$

with

$$\mu = \max_{t \in I} \beta(t) - \min_{t \in I} \alpha(t),$$

$$\nu = \frac{\max\{|\alpha(a) - \beta(b)|, |\alpha(b) - \beta(a)|\}}{b-a},$$

$$\|h\|_p = \begin{cases} \sup_{t \in I} |h(t)| & \text{if } p = \infty, \\ \left[\int_a^b |h(t)|^p dt\right]^{1/p} & \text{if } 1 \leq p < \infty, \end{cases}$$

and considering $(p-1)/p \equiv 1$ for $p = \infty$.

Assuming assumptions $(H1) - (H5)$, we fix $K \geq \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty\}$ and such that

$$\min\left\{\int_{\phi(\nu)}^{\phi(K)} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du, \int_{\phi(-K)}^{\phi(-\nu)} \frac{|\phi^{-1}(u)|^{\frac{p-1}{p}}}{\theta(|\phi^{-1}(u)|)} du\right\} > \mu^{\frac{p-1}{p}} \|h\|_p. \tag{2.3}$$

Analogously to the proof given in [8, Theorem 4.1,Lemma 2.3], we can prove the following existence theorem.

Theorem 2.1 *Assume hypotheses* $(H1) - (H5)$. *Then problem (2.1) has extremal solutions in* $[\alpha, \beta]$.

Moreover if $u \in [\alpha, \beta]$ *is a solution of (2.1) then* $|u'(t)| \leq K$ *for all* $t \in I$.

3 Existence results for ϕ − Laplacian equations

In this section we prove the existence of extremal solutions of problem (2.1) under weaker assumptions in the regularity of the nonlinear part of the equation. The proof follows from the generalized iterative techniques developed in [12]. We assume the following condition.

($\overline{\text{H1}}$) $f : I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies: the function $f(\cdot, u(\cdot), v)$ is measurable on I for all $v \in \mathbb{R}$, whenever $u : I \to \mathbb{R}$ is a continuous function; $f(t, u, \cdot)$ is continuous on $\mathbb{R}$ for a.a. $t \in I$ and for all $u \in \mathbb{R}$; for every $M > 0$ there exists a real-valued function $\psi_M \in L^1(I)$ such that for every $(u, v) \in \mathbb{R}^2$ with $|u| \leq M$ and $|v| \leq M$.

$$|f(t, u, v)| \leq \psi_M(t) \quad \text{for a.a. } t \in I.$$

Now we impose to function f the following condition.

(H6) There exists a L^1-Carathéodory function $g : I \times \mathbb{R} \to \mathbb{R}$ such that for a.a. $t \in I$ and for all $v \in \mathbb{R}$ we have

$$f(t, u_1, v) - f(t, u_2, v) \leq g(t, u_2) - g(t, u_1), \quad \text{for all } \alpha(t) \leq u_1 \leq u_2 \leq \beta(t).$$

It is clear, defining $g(t,x) \equiv M\,x$ for some $M > 0$, that such condition covers the classical one-sided Lipschitz condition that is usually imposed to develop the monotone method (see [8] and references therein). On the other hand, condition $(\overline{H1})$ allows us to consider nonlinearities discontinuous at the first two variables.

Now, we are in a position to prove the following existence result.

Theorem 3.1 *Assume hypotheses $(\overline{H1})$,(H2)–(H6). Then problem (2.1) has extremal solutions in $[\alpha,\beta]$.*

Proof. We are going to prove the existence of the minimal solution of (2.1). The existence of the maximal solution follows from the dual arguments.

For each $\eta \in [\alpha,\beta]$ consider the problem

$$(P_\eta) \quad \begin{cases} -[\bar{\phi}(u')]'(t) = F_\eta(t,u(t),u'(t)) \quad \text{a.a. } t \in I, \\ \\ L_1(u(a),u(b),u'(a),u'(b),u) = L_2(u(a),u(b)) = 0, \end{cases}$$

where $F_\eta : I \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$F_\eta(t,x,y) := f(t,\eta(t),\delta_K(y)) + g(t,\eta(t)) - g(t,x),$$

$$\bar{\phi}(x) = \begin{cases} x - K + \phi(K), & \text{for } x > K, \\ \phi(x), & \text{for } -K \le x \le K, \\ x + K + \phi(-K), & \text{for } x < -K, \end{cases} \tag{3.1}$$

and $\delta_K(y) := \min\{\max\{-K,y\},K\}$, where K is given in (2.3).

By hypotheses (H4) and (H6) and the definition of K we have that, for each $\eta \in [\alpha,\beta]$, the functions α and β are lower and upper solutions, respectively, for problem (P_η). Obviously, conditions $(H1)-(H4)$ are satisfied for problem (P_η). It is not difficult to verify that

$$|F_\eta(t,u,v)| \le (h(t)+2\,\psi(t))(\theta(|\delta_K(v)|)+1) \ \text{ for all } u \in [\alpha(t),\beta(t)] \text{ and all } v \in \mathbb{R}.$$

Here h and θ are given in $(H5)$ and ψ is introduced in $(H1)$ for $x \in [\alpha(t),\beta(t)]$ and $|y| \le K$. Note that all of them are independent on η.

Thus, the Nagumo's condition $(H5)$ is fulfilled for such functions. In consequence, there exist the extremal solutions in $[\alpha,\beta]$ of problem (P_η). Moreover, follow the arguments in [7, Lemma 2.3], we deduce that there exists $K^* > 0$, independent on η, such that $|u'| < K^*$ in I, for all u a solution of problem (P_η).

Now, we define the operator $G : [\alpha,\beta] \to [\alpha,\beta]$ as

$$G\eta := \text{ minimal solution in } [\alpha,\beta] \text{ of problem } (P_\eta).$$

Claim 1. $G : [\alpha,\beta] \to [\alpha,\beta]$ is nondecreasing.

Fix η, $\xi \in [\alpha,\beta]$, such that $\eta \le \xi$ in I. Then by condition (H6) we have for a.a. $t \in I$ and for all $(x,y) \in \mathbb{R}^2$ that

$$F_\eta(t,x,y) \le F_\xi(t,x,y).$$

Therefore, by the definition of G, it follows that $G\xi \in [\alpha,\beta]$ is an upper solution for problem (P_η). Hence Theorem 2.1 ensures that (P_η) has a solution in the interval $[\alpha, G\xi]$ and therefore $G\eta$, which is the minimal solution of (P_η) in $[\alpha,\beta]$, satisfies that $G\eta \le G\xi$.

Claim 2. G has the minimal fixed point x_* in $[\alpha, \beta]$.

By the definition of G, we know that

$$|(G\eta)'(t)| < K^* \quad \text{for all } t \in I.$$

Moreover since $G\eta \in AC(I)$ we have that the fixed points of G are the same that those of its restriction $\tilde{G} := G_{|_{[\alpha,\beta] \bigcap AC(I)}}$. Now, applying [12, Proposition 1.4.4], we have that $\tilde{G}$ has the extremal fixed points in $[\alpha, \beta] \bigcap AC(I)$ and, in particular, there exists x_* the least fixed point of G in $[\alpha, \beta]$, i.e., $x_* \in [\alpha, \beta]$, $x_* = Gx_*$ and if $x \in [\alpha, \beta]$ with $x = Gx$ then $x_* \leq x$. Moreover x_* satisfies

$$x_* = \min\{x \in X \ : \ Gx \leq x\}. \tag{3.2}$$

Claim 3. $x_* \in [\alpha, \beta]$ is the minimal solution of problem (2.1)

By the definition of G we have that

$$\begin{cases} -[\bar{\phi}(x_*')]'(t) = f(t, x_*(t), \delta_K(x_*'(t))) \quad \text{a.a. } t \in I, \\ \\ L_1(x_*(a), x_*(b), x_*'(a), x_*'(b), x_*) = L_2(x_*(a), x_*(b)) = 0. \end{cases}$$

Proceeding as in the proof of [7, Lemma 2.3] one can see that $|x_*'(t)| < K$ for all $t \in I$ and therefore x_* is a solution of (2.1).

Now suppose that $x \in [\alpha, \beta]$ is another solution of (2.1). Then by Theorem 2.1 we have that $|x'(t)| < K$ for all $t \in I$ and thus x is also a solution of problem (P_x). By the definition of G we have that $Gx \leq x$ and hence from (3.2) it follows that $x_* \leq x$. □

4 A ϕ − Laplacian − diffusion equation

In this section we consider a general formulation that includes ϕ − Laplacian and diffusion equations as a particular cases. We will see that, after a suitable change of variables, we can treat it as in the previous section. The considered problem is the following.

$$\begin{cases} -[\phi(k(u)u')]'(t) = f(t, u(t), k(u(t))u'(t)) \quad \text{a.a. } t \in I, \\ \\ L_1(u(a), u(b), k(u(a))u'(a), k(u(b))u'(b), u) = L_2(u(a), u(b)) = 0. \end{cases} \tag{4.1}$$

Now, we introduce the definitions of lower and upper solutions for this problem.

Definition 4.1 *We say that*

$$\alpha \in \mathcal{S}_2 := \{u \in \mathcal{C}^1(I) : k(u)u' \in \mathcal{C}(I), \ \phi(k(u)u') \in AC(I)\}$$

is a lower solution of problem (4.1) if satisfies

$$\begin{cases} -[\phi(k(\alpha)\alpha')]'(t) \leq f(t,\alpha(t),k(\alpha(t))\alpha'(t)) \quad a.a.\ t \in I, \\ \\ L_1(\alpha(a),\alpha(b),k(\alpha(a))\alpha'(a),k(\alpha(b))\alpha'(b),\alpha) \geq L_2(\alpha(a),\alpha(b)) = 0. \end{cases}$$

Analogously we say that $\beta \in \mathcal{S}_2$ is an upper solution of problem (2.1) if the above inequalities are reversed.

We say that $x \in \mathcal{S}_2$ is a solution of problem (2.1) if it is both a lower and an upper solution. The concept of extremal solutions is analogous to that given in Definition 2.1 for the problem (2.1).

We consider the following assumptions:

($\overline{\text{H4}}$) (i) There exist α, $\beta \in \mathcal{S}_2$ a lower and an upper solution of problem (4.1) respectively, such that $\alpha \leq \beta$ in I.

(ii) $k : \mathbb{R} \to \mathbb{R}$ is continuous and there exists $c > 0$ such that

$$k(u) > c \quad \text{for all } u \in \left[\min_{s\in I} \alpha(s), \max_{s\in I} \beta(s)\right].$$

($\overline{\text{H5}}$) $f : I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies (H5) with

$$\mu = \max_{t\in I} W(\beta(t)) - \min_{t\in I} W(\alpha(t)),$$

and

$$\nu = \frac{\max\{|W(\alpha(a)) - W(\beta(b))|, |W(\alpha(b)) - W(\beta(a))|\}}{b-a},$$

where $W(u) = \int_0^u k(s)ds$ for all $u \in \mathbb{R}$.

Next we prove the following existence result.

Theorem 4.1 *Assume hypotheses ($\overline{H1}$),(H2),(H3), ($\overline{H4}$), ($\overline{H5}$) and (H6). Then problem (4.1) has extremal solutions in $[\alpha, \beta]$.*

Proof. From the definition of W given in ($\overline{\text{H5}}$), it is clear by ($\overline{\text{H4}}$)-(ii) that $u \in \mathcal{S}_2 \bigcap [\alpha, \beta]$ is a solution of problem (4.1) if and only if $u \in \mathcal{S}_3 \bigcap [\alpha, \beta]$ is a solution of

$$\begin{cases} -[\phi((W \circ u)')]'(t) = f(t,u(t),(W \circ u)'(t)) \quad \text{a.a. } t \in I, \\ \\ L_1(u(a),u(b),(W \circ u)'(a),(W \circ u)'(b),u) = L_2(u(a),u(b)) = 0, \end{cases} \tag{4.2}$$

where

$$\mathcal{S}_3 := \{u : I \to \mathbb{R} : W \circ u \in \mathcal{C}^1(I),\ \phi \circ (W \circ u)' \in AC(I)\}.$$

Since W is an increasing homeomorphism, if we make the change of variable $v = W \circ u$, then $u \in \mathcal{S}_3$ is a solution of (4.2) if and only if $v \in \mathcal{S}_1$ is a solution of problem

$$\begin{cases} -[\bar{\phi}(v')]'(t) = \tilde{f}(t,v(t),v'(t)) \quad \text{a.a. } t \in I, \\ \\ \bar{L}_1(v(a),v(b),v'(a),v'(b),v) = \bar{L}_2(v(a),v(b)) = 0, \end{cases} \tag{4.3}$$

with

$$\tilde{f}(t,x,y) := f(t, W^{-1}(x), y),$$

$$\bar{L}_1(x,y,z,w,\xi) := L_1(W^{-1}(x), W^{-1}(y), z, w, W^{-1}\circ\xi),$$

and

$$\bar{L}_2(x,y) := L_2(W^{-1}(x), W^{-1}(y)).$$

We note that problem (4.3) is of the type (2.1). Moreover it follows that $\tilde{\alpha} := W \circ \alpha$ and $\tilde{\beta} := W \circ \beta$ are lower and upper solutions, respectively, of problem (4.3). On the other hand, since W^{-1} is increasing, using condition (H6) we have that for all $y \in \mathbb{R}$ and for all $\tilde{\alpha}(t) \leq x_1 \leq x_2 \leq \tilde{\beta}(t)$ for a.a. $t \in I$.

$$\tilde{f}(t,x_1,y) - \tilde{f}(t,x_2,y) \leq g(t, W^{-1}(x_2)) - g(t, W^{-1}(x_1)).$$

Then $\tilde{f}$ satisfies (H6) with the function $\tilde{g} : I \times \mathbb{R} \to \mathbb{R}$ defined as

$$\tilde{g}(t,x) := g(t, W^{-1}(x)).$$

Therefore, from our assumptions, it is easy to check that problem (4.3) satisfies hypotheses $(\overline{H1})$, (H2)-(H6) and thus Theorem 3.1 ensures the existence of extremal solutions in $[\tilde{\alpha}, \tilde{\beta}]$ of (4.3) which are the extremal solutions in $[\alpha, \beta]$ of (4.1). □

5 An example

In this section we present an example where Theorem 4.1 is applied. With this example we try to illustrate the kind of problems that can be studied under the formulation explained at the paper.

Let $r, s > 0$, $h \in L^1([0,1])$, $h \geq 0$ a. e. in $[0,1]$, $c > 0$, $J_1 \subset [0,1]$ a closed set and $J_2 \subset [0,1]$ a Lebesgue-measurable set. Consider the problem

$$\begin{cases} -\dfrac{d}{dt}\left(\dfrac{(1+\sin u(t))\,u'(t)|(1+\sin u(t))\,u'(t)|}{1+((1+\sin u(t))\,u'(t))^2}\right) = \\[2ex] \qquad = \dfrac{h(t)\,[s\,u(t)]\,u^r(t)|(1+\sin u(t))\,u'(t)|}{c}, \quad \text{for a. a. } t \in [0,1], \\[2ex] u(0) = 1/4\left(1 + \min\limits_{t\in J_1} u(t) + \displaystyle\int_{J_2} u^3(r)\,dr\right) = u(1). \end{cases}$$

Here $[\cdot]$ denotes the integer part. This problem follows the formulation of (4.1), defining

$$k(x) = 1 + \sin x \quad \text{for all } x \in \mathbb{R},$$

$$\phi(x) = \frac{x|x|}{1+x^2} \quad \text{for all } x \in \mathbb{R},$$

$$f(t,x,y) = \frac{h(t)\,[sx]\,x^r\,|y|}{c} \quad \text{for all } t \in [0,1] \text{ and all } x,\, y \in \mathbb{R},$$

$$L_1(x,y,z,w,\xi) = 1/4\left(1 + \min_{t\in J_1}\xi(t) + \int_{J_2}\xi^3(r)\,dr\right) - x,$$

for $(x,y,z,w,\xi) \in \mathbb{R}^4 \times \mathcal{C}(I)$ and

$$L_2(x,y) = y - x \quad \text{for } (x,y) \in \mathbb{R}^2.$$

It is easy to check that assumptions $(\overline{H1})$,(H2),(H3) and (H6) hold. Moreover condition $(\overline{H4})$ is satisfied for $\alpha = 0$ and $\beta = 1$, and condition $(\overline{H5})$ is also satisfied provided that

$$c > \frac{[s]\|h\|_1}{\frac{1}{2}\left(\frac{\pi}{2} - \frac{\nu}{1+\nu^2} - \arctan(\nu)\right)},$$

where $\nu = 2 - \cos(1)$. Therefore, for such values of c, Theorem 4.1 ensures the existence of extremal solutions in $[\alpha,\beta]$.

References

[1] Anderson, D. and Lisak, M., Approximate solutions of nonlinear diffusion equations, *Phys. Rev.* A 22 (1980), 2761 – 2768.

[2] Cabada, A., Cid, J. A. and Pouso R. L. Positive solutions for a class of singular differential equations arising in diffusion processes. *Dynam. Contin. Discr. Impuls. Syst.* (to appear).

[3] Cabada, A., Cid, J. A. and Pouso R. L. Existence Results and Approximation Methods for Functional Ordinary Differential Equations with Singular Diffusion–Type Differential Operators. *Comput. & Math. Appl.*, 44, (2002), 111 – 124.

[4] Cabada, A., Cid, J. A. and Pouso R. L. Positive Solutions for a Class of Implicit and Discontinuous Second Order Functional Differential Equations with Singularities. *Appl. Math. Comput.*, 137, (2003), 89 – 99.

[5] Cabada, A., Nieto, J. J. and Pouso, R. L., Approximate solutions to a new class of nonlinear diffusion problems, *J. Comput. Appl. Math.*, 108 (1999), 219 – 231.

[6] Cabada, A. and Heikkilä, S. Extremality Results for Discontinuous Explicit and Implicit Diffusion Problems. *J.Comput. Appl. Math.*, 143, (2002), 69 – 80.

[7] A. Cabada and R. L. Pouso, Existence results for the problem $(\phi(u'))' = f(t,u,u')$ with nonlinear boundary conditions, *Nonlinear Anal.* 35 (1998), 221-231.

[8] A. Cabada and R. L. Pouso, Extremal solutions of strongly nonlinear discontinuous second-order equations with nonlinear functional boundary conditions, *Nonlinear Anal.* 42 (2000), 1377-1396.

[9] A. Cabada and R. L. Pouso, Existence Theory for Functional p-Laplacian Equations with Variable Exponents. *Nonlinear Anal.*, 52, (2003), 557 – 572.

[10] M. Cherpion, C. De Coster and P. Habets, Monotone iterative methods for boundary value problems, *Differ. Integral Equat.*, 12, 3 (1999), 309-338.

[11] J.R. Esteban and J.L. Vázquez, On the equation of turbulent filtration in one-dimensional porous media, *Nonlinear Anal.*, 10 (1986), 1303-1325.

[12] S. Heikkilä and V. Lakshmikantham, "Monotone iterative techniques for discontinuous nonlinear differential equations", Marcel Dekker, New York (1994).

[13] M.A. Herrero and J.L. Vázquez, On the propagation properties of a nonlinear degenerate parabolic equation, *Comm. Partial Differential Equations,* **7** *(1982), 1381-1402.*

[14] Huang, Q. and Li, Y. Nagumo theorems of nonlinear singular boundary value problems. *Nonlinear Anal., Theory Methods Appl.* 29, 12, (1997), 1365-1372.

[15] King, J. R., Approximate solutions to a nonlinear diffusion equations, *J. Engrg. Math.*, 22 (1998), 53-72.

[16] Nieto, J. J. and Okrasinski, W., Existence, uniqueness and approximation of solutions to some nonlinear diffusion problems, *J. Math. Anal. Appl.,* 210 (1997), 231 – 240.

[17] Okrasinski, W., Integral equations methods in the theory of the water percolation, *Mathematical methods in fluid mechanics*, (Oberwolfach, 1981), *Methoden Verfahren Math. Phys.*, 24 (1982), 167 – 176.

[18] Okrasinski, W., On approximate solutions to some nonlinear diffusion problems, *Z. Angew. Math. Phys.*, 44 (1993), 722 – 731.

[19] D. O'Regan, Some general principles and results for $(\phi(y'))' = qf(t, y, y'),\ 0 < t < 1$, *SIAM J. Math. Anal.,* 24 (1993), 648 – 668.

[20] D. O'Regan, Existence theory for $(\phi(y'))' = f(t, y, y'),\ 0 < t < 1$, *Commun. Appl. Anal.,* 1 (1997), 33 – 52.

[21] J. Wang and W. Gao, Existence of solutions to boundary value problems for a nonlinear second order equation with weak Carathéodory functions, *Differential Equations Dynam. Systems,* 5, 2 (1997), 175 – 185.