LYAPUNOV STABILITY OF PERIODIC SOLUTIONS OF BRILLOUIN TYPE EQUATIONS

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Abstract. Motivated by the Brillouin equation we deal with the existence and Lyapunov stability of periodic solutions for a more general kind of equations. Our approach is based on the third order approximation in combination with some location information obtained by the averaging method. We will show that our main results apply to some singular models not previously covered in the related literature.

1. Introduction

During the past half century, the following classical Brillouin equation

$$\ddot{x} + \varepsilon(1 + \cos t)x = \frac{1}{x}, \quad \varepsilon > 0 \tag{1}$$

has been widely investigated because, as shown in [1], it governs a focusing system for an electron beam immersed in a periodic magnetic field. For the periodic solution of equation (1), a numerical conjecture is that (1) admits positive periodic solutions for $0 < \varepsilon < \frac{1}{4}$, see [16, Conjecture 6.1]. It seems that the first reference in this line is [7], where the existence of periodic solutions is proved when $\varepsilon \le \frac{1}{16}$ by using an analysis of the phase plane. Afterwards, this result has been subsequently improved up to $\varepsilon \le 0.2483\ldots$, see [2]. However, until now, the conjecture has not been proved by a strictly analytical method, [19], and in fact some doubts have been expressed about its validity, [2].

Compared with the existence results, only a few works focus on the stability of periodic solutions for (1). As far as we know, the first result along this line was proved in [14], in which the author shows that the solution given by Ding [7] is stable in the linear sense, although the Lyapunov stability of the solution is not assured since it may depend on the nonlinear terms. In [3, 15, 17], some stability results have been obtained for the related equation

$$\ddot{x} + \varepsilon(1 + \delta\cos t)x = \frac{1}{x}, \quad \epsilon > 0, \quad \delta > 0. \tag{2}$$

Unfortunately, those stability results cannot be applied to the model (1) and, to the best of our knowledge, nothing has been published concerning the Lyaupunov stability for the Brillouin equation (1).

Notice that equation (2) is a particular case of equation

$$\ddot{x} = r(t)x^{\alpha} - \varepsilon s(t)x^{\beta}, \tag{3}$$

where $\alpha, \beta \in \mathbb{R}$, and r, s are continuous T-periodic functions. The existence and stability of solutions for equation (3) have been studied, for $\alpha < \beta < 0$ in [5] and for $0 < \alpha < \beta < 1$ in

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[18], by application of the averaging method and the third-order approximation. However, a careful analysis of their proofs reveals that actually they remain valid for a wider set of exponents. In this way, we will be able not only to unify the results in both papers but also to obtain new applications to equations like (2), that are covered by none of them since one of the exponents is negative while the other being positive. We also notice that all the stable periodic solutions obtained in this paper are of twist type and so the complicated dynamics prescribed by the KAM theory appears around them, see [5].

Finally, throughout this paper, for a given T-periodic function e, we denote

$$e_m = \inf_{t\in[0,T]} e(t), \quad e_M = \sup_{t\in[0,T]} e(t) \quad \text{and} \quad \bar{e} = \frac{1}{T}\int_0^T e(t)dt.$$

2. Main results

2.1. Existence of periodic solutions for equation (3). The following result unifies and extends [5, Lemma 3.1] and [18, Theorem 3.3]. The proof is an application of the averaging method, see [8], and it is included for the sake of completeness.

Theorem 2.1. *Assume that r and s are T-periodic continuous functions with $\bar{r}\cdot\bar{s} > 0$. Let $\alpha, \beta \in \mathbb{R}$ be such that*

$$(\beta-\alpha)(1-\alpha) > 0. \tag{4}$$

Then equation (3) has a T-periodic solution $x(t,\varepsilon)$ if $\varepsilon > 0$ is small enough. Moreover, the following asymptotic behavior holds

$$\lim_{\varepsilon\to 0^+} \varepsilon^\gamma x(t,\varepsilon) = \omega^\gamma, \qquad \text{uniformly in } t, \tag{5}$$

where $\omega = \frac{\bar{r}}{\bar{s}}$ and $\gamma = \frac{1}{\beta-\alpha}$.

Proof. Firstly, we rewrite equation (3) as the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= r(t)x^\alpha - \varepsilon s(t)x^\beta, \end{aligned} \tag{6}$$

and by rescaling the variables $x = u\varepsilon^{-\gamma}$, $y = v\varepsilon^{-\frac{\gamma(\alpha+1)}{2}}$ and $\nu = \varepsilon^{\frac{\gamma(1-\alpha)}{2}}$, system (6) takes the form

$$\begin{aligned} \dot{u} &= \nu v, \\ \dot{v} &= \nu\left(r(t)u^\alpha - s(t)u^\beta\right). \end{aligned} \tag{7}$$

Notice that condition (4) implies that $\frac{\gamma(1-\alpha)}{2} > 0$ and then $\nu \to 0^+$ if and only if $\varepsilon \to 0^+$. The averaged system of (7) is

$$\begin{aligned} \dot{\xi} &= \nu\eta, \\ \dot{\eta} &= \nu\left(\bar{r}\xi^\alpha - \bar{s}\xi^\beta\right). \end{aligned} \tag{8}$$

Then (8) has a unique non-trivial constant solution $(\xi_0, \eta_0) := (\omega^\gamma, 0)$, and the Jacobian matrix evaluated at (ξ_0, η_0) is

$$M = \begin{pmatrix} 0 & \nu \\ \nu\left(\alpha\bar{r}\xi_0^{\alpha-1} - \beta\bar{s}\xi_0^{\beta-1}\right) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nu \\ \nu(\alpha-\beta)\bar{r}\left(\frac{\bar{r}}{\bar{s}}\right)^{\gamma(\alpha-1)} & 0 \end{pmatrix}. \tag{9}$$

Then (ξ_0, η_0) is nondegenerate since (4) implies in particular that $\alpha \neq \beta$. So, by [8, Section V.3], the equilibrium (ξ_0, η_0) is continuable for small ν, that is, there exists ν_0 such that system (7) has a T-periodic solution $(u(t,\nu), v(t,\nu))$ for $0 < \nu < \nu_0$, tending uniformly to (ξ_0, η_0) as $\nu \to 0^+$. Going back through the rescaling, we conclude that equation (3) has a T-periodic solution $x(t,\varepsilon)$ for $\varepsilon > 0$ small enough and the asymptotic behavior (5) occurs. ∎

2.2. **Instability of periodic solutions for equation (3).** The following result extends [11, Theorem 3.1] in some situations.

Theorem 2.2. *Under the assumptions of Theorem 2.1, if moreover*

$$(\beta-\alpha)\overline{r}<0, \tag{10}$$

then there exists an unstable T-periodic solution $x(t,\varepsilon)$ of (3) provided that $\varepsilon>0$ is small enough.

Proof. From the proof of Theorem 2.1 we know that the Jacobian matrix of the averaged system (8) at the equilibrium (ξ_0,η_0) is given by (9) and that (ξ_0,η_0) is nondegenerate. Now, from condition (10) we have that M has one positive eigenvalue and then by [8, Section V.3] equation (3) has an unstable periodic solution when ε is small enough. □

2.3. **Lyapunov stability of periodic solutions for equation (3).** The following result unifies and extends [5, Theorem 3.2] and [18, Theorem 3.9]. Its proof is based on the method of the third approximation and the twist coefficient, see [12, 20], and it is similar to that of both results. So, we are going to focus mainly on the differences and skip the repeated parts.

Theorem 2.3. *Assume that r,s are T-periodic continuous functions with $\overline{r}\cdot\overline{s}>0$. Let $\alpha,\beta\in\mathbb{R}$ be such that (4) and the following conditions are satisfied*

$$2\alpha^2+2\beta^2+7\alpha\beta-\alpha-\beta-1\neq 0, \tag{11}$$

$$(\omega\beta s-\alpha r)_m>0. \tag{12}$$

Then the T-periodic solution $x(t,\varepsilon)$ of (3) obtained in Theorem 2.1 is stable if $\varepsilon>0$ is small enough.

Proof. It is enough to show that for small enough $\varepsilon>0$ the first twist coefficient μ given by formula (A.12) in [16] is different from zero. The third-order approximation of (3) is

$$\ddot{x}+a(t)x+b(t)x^2+c(t)x^3+o(x^3)=0,$$

where

$$a(t)=\varepsilon\beta s(t)x(t)^{\beta-1}-\alpha r(t)x(t)^{\alpha-1}, \tag{13}$$

$$b(t)=\frac{1}{2}\left[\varepsilon\beta(\beta-1)s(t)x(t)^{\beta-2}-\alpha(\alpha-1)r(t)x(t)^{\alpha-2}\right], \tag{14}$$

and

$$c(t)=\frac{1}{6}\left[\varepsilon\beta(\beta-1)(\beta-2)s(t)x(t)^{\beta-3}-\alpha(\alpha-1)(\alpha-2)r(t)x(t)^{\alpha-3}\right]. \tag{15}$$

By using the asymptotic behavior (5) and the expressions (13)-(15), we have the following limits uniformly in t

$$\lim_{\varepsilon\to 0^+}\varepsilon^{\gamma(\alpha-1)}a(t)=\beta s(t)\omega^{\gamma(\beta-1)}-\alpha r(t)\omega^{\gamma(\alpha-1)}, \tag{16}$$

$$\lim_{\varepsilon\to 0^+}\varepsilon^{\gamma(\alpha-2)}b(t)=\frac{1}{2}\left[\beta(\beta-1)s(t)\omega^{\gamma(\beta-2)}-\alpha(\alpha-1)r(t)\omega^{\gamma(\alpha-2)}\right],$$

and

$$\lim_{\varepsilon\to 0^+}\varepsilon^{\gamma(\alpha-3)}c(t)=\frac{1}{6}\left[\beta(\beta-1)(\beta-2)s(t)\omega^{\gamma(\beta-3)}-\alpha(\alpha-1)(\alpha-2)r(t)\omega^{\gamma(\alpha-3)}\right].$$

From (16) and condition (12) it follows that for all $t\in[0,T]$

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\varepsilon^{\gamma(\alpha-1)}a(t)&=\beta s(t)\omega^{\gamma(\beta-1)}-\alpha r(t)\omega^{\gamma(\alpha-1)}\\&=\omega^{\gamma(\alpha-1)}(\omega\beta s(t)-\alpha r(t))>0,\end{aligned}$$

which implies that $a(t)>0$ if $\varepsilon>0$ is small enough and then also $\bar{a}>0$ for small enough $\varepsilon>0$.

Now, an application of [6, Corollary 4.1] gives

$$\lim_{\varepsilon\to 0^+} \varepsilon^{\frac{\gamma(\alpha-1)}{2}}\theta = T\sqrt{(\beta-\alpha)\,\overline{r}\,\omega^{\gamma(\alpha-1)}},$$

and

$$\lim_{\varepsilon\to 0^+} \frac{R(t)}{\varepsilon^{\frac{\gamma(\alpha-1)}{4}}} = \frac{1}{\sqrt[4]{(\beta-\alpha)\,\overline{r}\,\omega^{\gamma(\alpha-1)}}},$$

for $\varepsilon > 0$ small enough. Here $\theta = T\rho$, where ρ is the rotation number of the associated Hill's equation $\ddot{u} + a(t)u = 0$, and $R(t)$ appears in [16, Definition A.5]. Then, it follows from [10, Lemma 3.6] that the associated Hill's equation is elliptic and 4-elementary if $\varepsilon > 0$ is small enough. Now, following the same reasoning as in the proof of [18, Theorem 3.9] and avoiding the tedious computations we get

$$\lim_{\varepsilon\to 0^+} \frac{\mu}{\varepsilon^{2\gamma}} = \frac{T}{48\omega^{2\gamma}}(2\alpha^2 + 2\beta^2 + 7\alpha\beta - \alpha - \beta - 1).$$

Condition (11) means that $\lim_{\varepsilon\to 0^+} \frac{\mu}{\varepsilon^{2\gamma}} \neq 0$ and therefore the twist coefficient μ is non-zero when ε is small enough. □

Remark 1. Notice that in [5, 18] condition (12) appears with the inequality "≥" instead of ">". However, a thorough examination of the proof reveals that in fact the strict inequality is required.

On the other hand, condition (12) is equivalent to $\omega\beta s(t) - \alpha r(t) > 0$, for all $t \in [0, T]$ and then dividing by T and integrating both sides of the inequality between 0 and T we get

$$(\beta - \alpha)\overline{r} > 0. \tag{17}$$

Observe that (17) is opposite to (10) and a necessary condition to apply Theorem 2.3.

Corollary 1. *Suppose that r and s are T-periodic continuous functions with $\overline{r}\cdot\overline{s} > 0$ and $\alpha, \beta \in \mathbb{R}$ satisfy (11), (12) and moreover*

$$\text{either} \quad \alpha < \beta < 1 \quad \text{or} \quad 1 < \beta < \alpha. \tag{18}$$

Then the equation

$$\ddot{x} = \lambda r(t)x^\alpha - s(t)x^\beta \tag{19}$$

has a stable T-periodic solution $x(t, \lambda)$ if λ is large enough.

Proof. By using the change of variables $x = \lambda^{\frac{1}{1-\alpha}}y$, equation (19) is transformed into $\ddot{y} = r(t)y^\alpha - \varepsilon s(t)y^\beta$, where $\varepsilon = \lambda^{\frac{\beta-1}{1-\alpha}}$. From (18) it follows that $\varepsilon \to 0^+$ if and only if $\lambda \to +\infty$. Note also that (18) implies (4) and then the result follows from Theorem 2.3. □

3. Applications to some singular models

3.1. **The Brillouin equation.** In this section, we will use our main mathematical results to make more complete the study of the singular Brillouin type differential equation

$$\ddot{x} = \frac{r(t)}{x^\sigma} - \varepsilon s(t)x. \tag{20}$$

Theorem 3.1. *Assume that r, s are T-periodic continuous functions with $\overline{r} > 0$ and $\overline{s} > 0$. Let $\sigma > 0$ be such that the following conditions are satisfied:*

$$\sigma \neq 3, \tag{21}$$

$$(\overline{r}\cdot s + \sigma\cdot\overline{s}\cdot r)_m > 0. \tag{22}$$

Then, equation (20) has a T-periodic solution $x(t, \varepsilon)$ stable in the Lyapunov sense if ε is small enough.

Proof. Since equation (20) is a particular form of (3) with $\alpha = -\sigma < 0$ and $\beta = 1$ the result follows from Theorem 2.3. □

Remark 2. The situation excluded by (21), that is $\sigma = 3$, corresponds to the Ermakov-Pinney equation [13] to which our results cannot be applied.

Corollary 2. *For any $\delta > 0$, the equation (2) has a 2π-periodic solution if ε is small enough. Furthermore, this 2π-periodic solution is stable in the sense of Lyapunov for ε small enough and $0 < \delta < 2$.*

Proof. The existence part is consequence of Theorem 2.1. The stability part follows from Theorem 3.1 with $r(t) \equiv 1$, $s(t) = 1 + \delta \cos t$ and $\sigma = 1$ (in this case, $\overline{r} = \overline{s} = 1$ and then $(\overline{r} \cdot s + \sigma \cdot \overline{s} \cdot r)_m = 2 - \delta$). □

As a relevant application of Corollary 2, we consider the case $\delta = 1$.

Corollary 3. *The classical Brillouin equation (1) has at least one Lyapunov stable 2π-periodic solution when ε is small enough.*

3.2. **A Rayleigh-Plesset equation.** The equation

$$\ddot{x} = \frac{\lambda}{x^{\frac{6k-1}{5}}} - s(t)x^{1/5} \tag{23}$$

governs the radial oscillations of a bubble in a liquid under the action of a radial pressure field if we neglect surface tension and viscosity but we consider the effect of the internal gas pressure (see [16, Chapter 9]). It has been proved in [9] that (23) has a periodic positive solution if $k \geq 1$ and $\bar{s} > 0$. As consequence of our main results we can add stability information, thus giving a partial answer to the Open Problem 9.1 in [16].

Theorem 3.2. *Assume that s is a T-periodic continuous function with $\overline{s} > 0$ and moreover $k > 0$, $k \neq \frac{3+\sqrt{57}}{12} \approx 0.879153$ and $s_m + (6k-1)\overline{s} > 0$. Then equation (23) has a T-periodic solution $x(t, \lambda)$ stable in the Lyapunov sense if $\lambda > 0$ is large enough.*

Proof. Note that equation (23) is a particular form of (19) with $r(t) = 1$, $\alpha = \frac{1-6k}{5}$ and $\beta = 1/5$. Then conditions (11), (12) and (18) are satisfied and the result follows from Corollary 1. □

3.3. **A Gylden-Meshcherskii type equation.** When $\beta = -2$ the equation

$$\ddot{x} = \frac{\mu^2}{x^3} - s(t)x^{\beta} \tag{24}$$

rules the radial component of a solution with angular moment μ of the Gylden-Meshcherskii problem, that is, a two-body problem with a periodically variable product of masses s (see [16, Chapter 4]). Our following result extends [4, Theorem 3.3] to a slightly wider set for the parameter β.

Theorem 3.3. *Assume that s is a T-periodic continuous function with $\overline{s} > 0$, $-3 < \beta < 1$ and*

$$(\beta s)_m + 3\overline{s} > 0. \tag{25}$$

Then equation (24) has a T-periodic solution $x(t, \mu)$ stable in the Lyapunov sense if $\mu > 0$ is large enough.

Proof. Note that equation (24) is a particular form of (19) with $r(t) = 1$ and $\alpha = -3$. Then conditions (11), (12) and (18) are satisfied and the result follows from Corollary 1. □

Remark 3. In the original Gylden-Meshcherskii problem, that is when $\beta = -2$, condition (25) reads as

$$\frac{s_M}{\overline{s}} < \frac{3}{2}. \tag{26}$$

We point out that condition (26) is not explicitly asked in [4, Theorem 3.3] but a close inspection of its proof shows that in fact it should be assumed.

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