On uniqueness criteria for systems of ordinary differential equations

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Abstract

We present some new uniqueness criteria for the Cauchy problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

based on the local equivalence with another initial value problem.

Keywords: Uniqueness criteria ; Cauchy problem ; Ordinary differential equations.

1 Introduction

The question about existence and uniqueness of solution for the initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f : U \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, U is an open set and $(t_0, x_0) \in U$, is a classical problem in the study of differential equations and it has a great importance, as much in theory as in applications.

In spite of the enormous literature that exists about this topic and the great amount of sufficient conditions that imply uniqueness of solutions (see [1, 5] and the references therein), this problem is far from being completely solved. When (1.1) is scalar and autonomous we have sufficient and necessary conditions for existence and uniqueness of solution (see [2]). However, for the general case we know no condition on f being at the same time necessary and sufficient. Neither it seems easy to find them, as it is shown in theorem 2, chapter 12 in [6], there exists a Lebesgue nonmeasurable function $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that the Cauchy problem

$$x' = \Psi(t, x), \quad x(t_0) = x_0,$$

has a unique solution $x : \mathbb{R} \to \mathbb{R}$ (locally absolutely continuous) through any point $(t_0, x_0) \in \mathbb{R}^2$.

From now on we will center our attention in problem (1.1) with a continuous right-hand side f. In this case Peano's theorem ensures the existence of at least one solution, but it is easy to give examples where uniqueness fails: $x' = x^{\frac{2}{3}}$ has infinitely many solutions through $(0, 0)$. A more complicated example is given by Hartman in [5], page 18, where a scalar and continuous function f in $\mathbb{R}^2$ is defined in such a way that there is more than one solution of problem (1.1) for every initial condition. A remarkable result, related with uniqueness, is that for almost each (in the category sense) function in the Banach space of all bounded and continuous real functions defined in $\mathbb{R}^2$, problem (1.1) has a unique solution (see theorem 1, chapter 12 in [6]). Nevertheless, like in the discontinuous case, there is no characterization for uniqueness of solution. The most important uniqueness criterion in the case of continuous f's is that of Lipschitz. This classic result was given in 1876 (see [7]). Briefly, we say that $f : U \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with respect to x, if for every $(t_0, x_0) \in U$ there exists a neighbourhood $V \subset U$ of (t_0, x_0) and a constant $K > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2| \quad \text{for all } (t, x_1), (t, x_2) \in V.$$

The Lipschitz criterion, as it is well-known, says that if f is continuous and locally Lipschitz continuous with respect to x, then the problem (1.1) has a unique local solution. On the other hand, it is easy to prove that the Lipschitz criterion is not necessary: for example, in the following initial value problem (example 1.2.2 in [1])

$$x' = f(t,x) = 1 + x^{\frac{2}{3}}, \quad x(0) = 0, \tag{1.2}$$

the function f is continuous in $\mathbb{R}^2$ and it is not Lipschitz continuous in any neighbourhood of $(0,0)$. However, separating the variables and using the substitution $x = z^3$, we deduce that the unique solution of problem (1.2) is implicitly given by the equation

$$3(x^{\frac{1}{3}} - \arctan(x^{\frac{1}{3}})) = t.$$

In [3] we prove the following alternative version of Lipschitz criterion for the scalar case: if $f : U \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in the open set U, $(t_0, x_0) \in U$, $f(t_0, x_0) \neq 0$ and f is locally Lipschitz continuous with respect to t, then problem (1.1) has a unique local solution. This surprising result can be applied, for example, to the problem (1.2) for which the usual Lipschitz criterion fails.

The present paper is organized as follows: in section 2 we establish a local equivalence between two initial value problems, which will be fundamental in the proof of our results. In section 3 we generalize the version of Lipschitz uniqueness criterion given in [3] to systems of differential equations. Some corollaries and examples are also given. In section 4 we prove that continuity implies local uniqueness for a class of two dimensional autonomous systems, which includes the Hamiltonian ones, provided that the initial condition is not a critical point. This result is closed related to that of [10].

2 Two (locally) equivalent IVP's

Let $U \subset \mathbb{R}^{n+1}$ be an open set, $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ a continuous function and $(t_0, x_0) \in U$. We consider the initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0. \tag{2.3}$$

We say that a function $x : I_x \to \mathbb{R}^n$ is a *solution* of problem (2.3) if satisfies the following conditions:

(i) $I_x \subset \mathbb{R}$ is an interval (not necessarily open) with non empty interior such that $t_0 \in I_x$;

(ii) for all $t \in I_x$, $(t, x(t)) \in U$;

(iii) for all $t \in I_x$ there exists $x'(t)$ and $x'(t) = f(t, x(t))$;

(iv) $x(t_0) = x_0$.

We point out that from (iii) it follows that x belongs to $\mathcal{C}^1$.

As usual, we define the norm $\|\cdot\|_\infty : \mathbb{R}^N \to [0, \infty)$, for $N \in \mathbb{N}$, as

$$\|(x_1, x_2, \ldots, x_N)\|_\infty := \max_{i \in \{1,2,\ldots,N\}} |x_i|,$$

and the open ball with center $x \in \mathbb{R}^N$ and radius $r > 0$ as

$$B_\infty(x, r) := \{y \in \mathbb{R}^N \ : \ \|x - y\|_\infty < r\}.$$

Since $f := (f_1, f_2, \ldots, f_{n-1}, f_n)$ is continuous, when $f_n(t_0, x_0) \neq 0$ there exist open intervals $J_i \subset \mathbb{R}$, with $i \in \{0, 1, \ldots, n\}$, such that setting $B = J_1 \times J_2 \times \cdots \times J_{n-1} \times J_n$ we have that

1. $(t_0, x_0) \in J_0 \times B \subset U$;
2. $f_n(t, x) \neq 0$ for all $(t, x) \in J_0 \times B$.

Then we can define $\tilde{f} : J_n \times \tilde{B} \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$, where $\tilde{B} = J_1 \times J_2 \times \cdots \times J_{n-1} \times J_0$, as

$$\tilde{f}_i(r, y_1, \ldots, y_{n-1}, y_n) = \frac{f_i(y_n, y_1, \ldots, y_{n-1}, r)}{f_n(y_n, y_1, \ldots, y_{n-1}, r)} \quad \text{if } i \in \{1, 2, \ldots, n-1\}$$

and

$$\tilde{f}_n(r, y_1, \ldots, y_{n-1}, y_n) = \frac{1}{f_n(y_n, y_1, \ldots, y_{n-1}, r)}.$$

Moreover, if $x_0 = (x_0^1, \dots, x_0^n) \in \mathbb{R}^n$ we define

$$r_0 = x_0^n \in J_n \quad \text{and} \quad y_0 = (x_0^1, x_0^2, \dots, x_0^{n-1}, t_0) \in \tilde{B},$$

and we consider the initial value problem

$$y'(r) = \tilde{f}(r, y(r)), \quad y(r_0) = y_0. \tag{2.4}$$

The proof of the following result is based on the chain rule and the formula for the derivative of the inverse.

Theorem 2.1 *Let $f : J_0 \times B \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a continuous function, $(t_0, x_0) \in J_0 \times B$ and $f_n(t, x) \neq 0$ for all $(t, x) \in J_0 \times B$.*
I) If $x : J_x \subset J_0 \to B$, with $x = (x_1, x_2, \dots, x_n)$, is a solution of the problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

then $y : x_n(J_x) \subset J_n \to \tilde{B}$ given by $y = (x_1 \circ x_n^{-1}, \dots, x_{n-1} \circ x_n^{-1}, x_n^{-1})$ is a solution of

$$y'(r) = \tilde{f}(r, y(r)), \quad y(r_0) = y_0.$$

II) Conversely, if $y : J_y \subset J_n \to \tilde{B}$, with $y = (y_1, y_2, \dots, y_n)$, is a solution of the problem

$$y'(r) = \tilde{f}(r, y(r)), \quad y(r_0) = y_0,$$

then $x : y_n(J_y) \subset J_0 \to B$ given by $x = (y_1 \circ y_n^{-1}, \dots, y_{n-1} \circ y_n^{-1}, y_n^{-1})$ is a solution of

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0.$$

In the following example we illustrate the theorem 2.1

Example 2.1 *We consider the function $f : \mathbb{R} \times (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ given by*

$$f(t, x_1, x_2) = \quad \frac{-t}{x_1}, \frac{1}{x_1}$$

and the problem

$$\begin{cases} x_1' = f_1(t, x_1, x_2) = \dfrac{-t}{x_1} \quad , \quad x_1(0) = 1, \\ \\ x_2' = f_2(t, x_1, x_2) = \dfrac{1}{x_1} \quad , \quad x_2(0) = 0. \end{cases} \tag{2.5}$$

Since $f_2(t,x_1,x_2) = \frac{1}{x_1} \neq 0$ *for all* $(t,x_1,x_2) \in \mathbb{R} \times (0,\infty) \times \mathbb{R}$, *we can define* $\tilde{f} : \mathbb{R} \times (0,\infty) \times \mathbb{R} \to \mathbb{R}^2$ *by*

$$\tilde{f}(r,y_1,y_2) = \left(\frac{f_1(y_2,y_1,r)}{f_2(y_2,y_1,r)}, \frac{1}{f_2(y_2,y_1,r)} \right) = (-y_2, y_1).$$

Hence, as $(y_1(t), y_2(t)) = (\cos(t), \sin(t))$ *for all* $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *is a solution of*

$$\begin{cases} y_1' = \tilde{f}_1(r,y_1,y_2) = -y_2 \quad , \quad y_1(0) = 1, \\ \\ y_2' = \tilde{f}_2(r,y_1,y_2) = y_1 \quad , \quad y_2(0) = 0, \end{cases}$$

it follows from theorem 2.1 that for all $t \in \sin(-\frac{\pi}{2}, \frac{\pi}{2}) = (-1,1)$

$$(x_1(t), x_2(t)) = (\cos(\arcsin t), \arcsin t),$$

defines a solution of (2.5).

Next, we establish that the local uniqueness for problem (2.4) implies the local uniqueness for problem (2.3). This result is fundamental in the following sections.

Theorem 2.2 *Let* $U \subset \mathbb{R}^{n+1}$ *be an open set,* $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ *a continuous function and* $(t_0, x_0) \in U$ *such that* $f_n(t_0,x_0) \neq 0$.

If there exists $\tilde{\alpha} > 0$ *such that the problem (2.4) has a unique solution on the interval* $[r_0 - \tilde{\alpha}, r_0 + \tilde{\alpha}]$ *then there exists* $\alpha > 0$ *such that the problem (2.3) has a unique solution on the interval* $[t_0 - \alpha, t_0 + \alpha]$.

Proof. We can suppose without loss of generality that $f_n(t_0,x_0) > 0$. Since f is continuous, $f_n(t_0,x_0) > 0$ and U is open there exist constants $a, b, m, M > 0$ such that

i) $J_0 \times B \subset U$, where $J_0 := (t_0 - a, t_0 + a)$ and $B := J_1 \times J_2 \times \ldots \times J_n$, with $J_i := (x_0^i - b, x_0^i + b)$ for $i \in \{1, 2, \ldots, n\}$;

ii) $0 < m \leq f_n(t,x)$ for all $(t,x) \in J_0 \times B$;

iii) $\|f(t,x)\|_\infty \leq M$ for all $(t,x) \in J_0 \times B$.

By hypothesis there exists a unique solution $y : [r_0 - \tilde{\alpha}, r_0 + \tilde{\alpha}] \to \tilde{B}$ of (2.4). Moreover, y_n is continuous, increasing and $y_n(r_0) = t_0$. Then, there exists $\alpha_1 > 0$ such that $[t_0 - \alpha_1, t_0 + \alpha_1] \subset y_n([r_0 - \tilde{\alpha}, r_0 + \tilde{\alpha}])$. We take $0 < \alpha < \min\{\alpha_1, \frac{b}{M}, \frac{\tilde{\alpha}}{M}\}$.

Existence of a solution on the interval $[t_0 - \alpha, t_0 + \alpha]$. By part (II) of theorem 2.1 the function $x : y_n([r_0 - \tilde{\alpha}, r_0 + \tilde{\alpha}]) \to B$, defined as $x = (y_1 \circ y_n^{-1}, \dots, y_{n-1} \circ y_n^{-1}, y_n^{-1})$ is a solution of (2.3). Moreover $[t_0 - \alpha, t_0 + \alpha] \subset y_n([r_0 - \tilde{\alpha}, r_0 + \tilde{\alpha}])$ and therefore x is a solution of (2.3) in $[t_0 - \alpha, t_0 + \alpha]$.

Uniqueness of solution on the interval $[t_0 - \alpha, t_0 + \alpha]$. Suppose that $\bar{x} : I_{\bar{x}} \to \mathbb{R}^n$ is a solution of (2.3) with $I_{\bar{x}} \subset [t_0 - \alpha, t_0 + \alpha]$. We will prove that $\bar{x}(I_{\bar{x}}) \subset B$ and $\bar{x}_n(I_{\bar{x}}) \subset (r_0 - \tilde{\alpha}, r_0 + \tilde{\alpha})$. Indeed, define $b_1 := \min\{b, \tilde{\alpha}\}$ and suppose that there exists $t_1 \in I_{\bar{x}}$ such that $\|\bar{x}(t_1) - x_0\|_\infty \geq b_1$ (we suppose that $t_1 > t_0$; the case $t_1 < t_0$ can be treated analogously). Put $t_2 := \inf\{t \in (t_0, t_1] \ : \ \|\bar{x}(t) - x_0\|_\infty = b_1\}$. It is obvious that for all $t \in (t_0, t_2)$ we have that $\bar{x}(t) \in B_\infty(x_0, b)$ and then $\|f(t, \bar{x}(t))\|_\infty \leq M$ for all $t \in (t_0, t_2)$. Hence

$$\|\bar{x}(t_2) - x_0\|_\infty = \left\| \int_{t_0}^{t_2} f(s, \bar{x}(s)) ds \right\|_\infty \leq \int_{t_0}^{t_2} \|f(s, \bar{x}(s))\|_\infty \, ds \leq M\alpha < b_1,$$

in contradiction with the definition of t_2.

Then, since $\bar{x} : I_{\bar{x}} \to \mathbb{R}^n$ is a solution of (2.3) and $\bar{x}(I_{\bar{x}}) \subset B$, by part (I) of theorem 2.1 we have that the function $\bar{y} : \bar{x}_n(I_{\bar{x}}) \to \tilde{B}$, defined as $\bar{y} = (\bar{x}_1 \circ \bar{x}_n^{-1}, \dots, \bar{x}_{n-1} \circ \bar{x}_n^{-1}, \bar{x}_n^{-1})$ is a solution of (2.4). Moreover, $\bar{x}_n(I_{\bar{x}}) \subset (r_0 - \tilde{\alpha}, r_0 + \tilde{\alpha})$ and then we have that $y(r) = \bar{y}(r)$ for all $r \in \bar{x}_n(I_{\bar{x}})$. Therefore, for all $t \in I_{\bar{x}}$ we have that

$$\begin{aligned} \bar{x}(t) &= (\bar{y}_1 \circ \bar{y}_n^{-1}(t), \dots, \bar{y}_{n-1} \circ \bar{y}_n^{-1}(t), \bar{y}_n^{-1}(t)) \\ &= (y_1 \circ y_n^{-1}(t), \dots, y_{n-1} \circ y_n^{-1}(t), y_n^{-1}(t)) = x(t). \end{aligned}$$

□

3 An alternative version of Lipschitz uniqueness criterion

We say that $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is *Lipschitz continuous when fixing component* $i_0 \in \{0, 1, \dots, n\}$ if there exists $K > 0$ such that

$$\|f(u_0, \dots, v_{i_0}, \dots, u_n) - f(\bar{u}_0, \dots, v_{i_0}, \dots, \bar{u}_n)\|_\infty \leq$$

$$K\|(u_0,\dots,u_{i_0-1},u_{i_0+1},\dots,u_n)-(\bar u_0,\dots,\bar u_{i_0-1},\bar u_{i_0+1},\dots,\bar u_n)\|_\infty,$$

for all $(u_0,\dots,v_{i_0},\dots,u_n),(\bar u_0,\dots,v_{i_0},\dots,\bar u_n)\in U$ and K is called a Lipschitz constant.

We say that f is *locally Lipschitz continuous when fixing component* $i_0\in\{0,1,\dots,n\}$ if for every $(t,x)\in U$ there exists a neighbourhood $V\subset U$ of (t,x) such that the restriction of f to V is Lipschitz continuous when fixing component $i_0\in\{0,1,\dots,n\}$.

We say that f is *(locally) Lipschitz continuous with respect to* x if it is (locally) Lipschitz continuous when fixing component $i_0=0$.

It is well-known that if there exists $\frac{\partial f(t,x)}{\partial x}$ and it is continuous in U, then f is locally Lipschitz continuous with respect to x. An analogous result, of course, is valid for the case of functions locally Lipschitz continuous when fixing a component i_0.

Now, we recall the classical Lipschitz criterion for the existence and local uniqueness of solutions for problem (2.3). It can be found in [4, 5, 8].

Theorem 3.1 *Let $U\subset\mathbb{R}^{n+1}$ be an open set, $f:U\subset\mathbb{R}^{n+1}\to\mathbb{R}^n$ and $(t_0,x_0)\in U$. We suppose that f is continuous and locally Lipschitz continuous with respect to x.*

Then there exists $\alpha>0$ such that the problem (2.3) has a unique solution in $[t_0-\alpha,t_0+\alpha]$.

Next, we present the main result of this section.

Theorem 3.2 *Let $U\subset\mathbb{R}^{n+1}$ be an open set, $f:U\subset\mathbb{R}^{n+1}\to\mathbb{R}^n$ and $(t_0,x_0)\in U$. We suppose that f is continuous and locally Lipschitz continuous when fixing a component $i_0\in\{0,1,\dots,n\}$.*

Then there exists $\alpha>0$ such that the problem (2.3) has a unique solution in $[t_0-\alpha,t_0+\alpha]$ provided that either $i_0=0$ or $f_{i_0}(t_0,x_0)\neq 0$.

Proof. If $i_0=0$ theorem 3.2 reduces to theorem 3.1. When $i_0\neq 0$ we can suppose without loss of generality that $i_0=n$ and $f_n(t_0,x_0)>0$. Then there exist open intervals $J_i\subset\mathbb{R}$, with $i\in\{0,1,\dots,n\}$, and constants $m,M>0$ such that

1. $(t_0, x_0) \in J_0 \times B \subset U$;

2. $f_n(t, x) \neq 0$ for all $(t, x) \in J_0 \times B$;

3. $0 < m \leq f_n(t, x)$ for all $(t, x) \in J_0 \times B$;

4. $\|f(t, x)\|_\infty \leq M$ for all $(t, x) \in J_0 \times B$;

5. $f : J_0 \times B \to \mathbb{R}^n$ is Lipschitz continuous in $J_0 \times B$ when fixing component $i_0 = n$, with Lipschitz constant $K > 0$;

where $B = J_1 \times J_2 \times \cdots \times J_{n-1} \times J_n$. We will prove that $\tilde{f} : J_n \times \tilde{B} \to \mathbb{R}^n$ is Lipschitz continuous with respect to $y \in \tilde{B}$. Indeed, if $i \in \{1, 2, \ldots, n-1\}$ we have that

$$\begin{aligned}
|\tilde{f}_i(r, y_1, \ldots, y_n) - \tilde{f}_i(r, \bar{y}_1, \ldots, \bar{y}_n)| &= \frac{f_i(y_n, y_1, \ldots, r)}{f_n(y_n, y_1, \ldots, r)} - \frac{f_i(\bar{y}_n, \bar{y}_1, \ldots, r)}{f_n(\bar{y}_n, \bar{y}_1, \ldots, r)} \\
&\leq \frac{2MK\|(y_1, \ldots, y_n) - (\bar{y}_1, \ldots, \bar{y}_n)\|_\infty}{m^2}.
\end{aligned}$$

On the other hand, if $i = n$ we have that

$$\begin{aligned}
|\tilde{f}_n(r, y_1, \ldots, y_n) - \tilde{f}_n(r, \bar{y}_1, \ldots, \bar{y}_n)| &= \frac{1}{f_n(y_n, y_1, \ldots, r)} - \frac{1}{f_n(\bar{y}_n, \bar{y}_1, \ldots, r)} \\
&\leq \frac{K\|(y_1, \ldots, y_n) - (\bar{y}_1, \ldots, \bar{y}_n)\|_\infty}{m^2}.
\end{aligned}$$

Then, taking $\tilde{K} := \max\{\frac{2MK}{m^2}, \frac{K}{m^2}\}$, we obtain that $\tilde{f} : J_n \times \tilde{B} \to \mathbb{R}^n$ is Lipschitz continuous with respect to $y \in \tilde{B}$. Therefore from theorem 3.1 and theorem 2.2 it follows the existence of a constant $\alpha > 0$ such that the problem (2.3) has a unique solution in the interval $[t_0 - \alpha, t_0 + \alpha]$. $\square$

Example 3.1 *The following autonomous initial value problem*

$$\begin{cases} x_1' = 1 & , \quad x_1(0) = 1, \\ \\ x_2' = \sqrt{|x_2|} & , \quad x_2(0) = 0, \end{cases}$$

has infinitely many solutions. We have that f is continuous in $\mathbb{R}^2$ and locally Lipschitz continuous when fixing component $i_0 = 2$, but $f_2(x_1, x_2) = \sqrt{|x_2|}$ vanishes at the initial condition $(1, 0)$. On the other hand, $f_1(x_1, x_2) = 1 \neq 0$ but f is not locally Lipschitz continuous when fixing component $i_0 = 1$.

Remark 3.1 *In theorem 3.2 we use the Lipschitz criterion to ensure local uniqueness for problem (2.4). We have chosen Lipschitz criterion for clarity and simplicity, but in an analogous way we can adapt other more general criteria: Osgood, Nagumo, Perron, Kamke... As an example we are going to give an alternative version of Osgood's criterion.*

Osgood's criterion: *(see theorem 1.4.2 in [1]) Let* $U \subset \mathbb{R}^2$ *be an open set,* $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ *and* $(t_0, x_0) \in U$. *We suppose that* f *is continuous and it satisfies that*

$$|f(t, x_1) - f(t, x_2)| \leq g(|x_1 - x_2|) \quad \textit{for all } (t, x_1), (t, x_2) \in U,$$

where $g : [0, \infty) \to \mathbb{R}$ *is continuous, nondecreasing,* $g(0) = 0$, $g(z) > 0$ *if* $z > 0$ *and* $\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{dz}{g(z)} = \infty$.

Then there exists $\alpha > 0$ *such that the problem (2.3) has a unique solution in* $[t_0 - \alpha, t_0 + \alpha]$.

When f *is scalar, Lipschitz criterion is a particular case of Osgood's (taking* $g(z) = Kz$*). Therefore, for a scalar* f *the following theorem is more general than theorem 3.2.*

Alternative version of Osgood's criterion: *Let* $U \subset \mathbb{R}^2$ *be an open set,* $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ *and* $(t_0, x_0) \in U$. *We suppose that* f *is continuous,* $f(t_0, x_0) \neq 0$ *and*

$$|f(t_1, x) - f(t_2, x)| \leq g(|t_1 - t_2|) \quad \textit{for all } (t_1, x), (t_2, x) \in U,$$

where $g : [0, \infty) \to \mathbb{R}$ *is continuous, nondecreasing,* $g(0) = 0$, $g(z) > 0$ *if* $z > 0$ *and* $\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{dz}{g(z)} = \infty$.

Then there exists $\alpha > 0$ *such that the problem (2.3) has a unique solution in* $[t_0 - \alpha, t_0 + \alpha]$.

3.1 Some consequences of theorem 3.2

We are going to give some corollaries and particular cases of theorem 3.2.

3.1.1 Global existence results.

Corollary 3.3 *Let $U \subset \mathbb{R}^{n+1}$ be an open set and $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$. We suppose that f is continuous and for each $(t,x) \in U$ there exist a component $i_0 = i_0(t,x) \in \{0,1,\ldots,n\}$ and a real number $b = b(t,x) > 0$ such that f restricted to $B_\infty((t,x),b)$ is Lipschitz continuous when fixing component i_0 and either $i_0 = 0$ or $f_{i_0}(t,x) \neq 0$.*

Then, given $(t_0,x_0) \in U$ there exists a unique maximal solution $x : J \to \mathbb{R}^n$ of problem (2.3), that is, if $z : I \to \mathbb{R}^n$ is also a solution of (2.3) trough (t_0,x_0), then $I \subset J$ and $x(t) = z(t)$ for all $t \in I$.

Proof. By theorem 3.2 problem (2.3) has an unique local solution trough any initial condition $(t,x) \in U$. Then, standard arguments (see theorem 5.6 in [8]) imply that for a fixed initial condition $(t_0,x_0) \in U$ there exists a unique maximal solution $x : J \to \mathbb{R}^n$ of problem (2.3). □

Example 3.2 *We consider the function*

$$f(t,x) = \begin{cases} \mathrm{e}^{\sqrt{x}} + t^3 \sin x & , \textit{if} \quad x \geq 0, \\ \\ x \ln(t^2+1) + \cos x & , \textit{if} \quad x < 0. \end{cases}$$

It is easy to check that f is locally Lipschitz continuous with respect to x in $\mathbb{R}^2 \backslash \{(t,0)\ :\ t \in \mathbb{R}\}$ and that it is not Lipschitz continuous with respect to x in any neighbourhood of $(t_0,0)$, with $t_0 \in \mathbb{R}$. Nevertheless $f(t_0,0) = 1 \neq 0$ and moreover f is Lipschitz continuous when fixing component $i_0 = 1$ in every bounded neighbourhood of $(t_0,0)$, for all $t_0 \in \mathbb{R}$. Hence, corollary 3.3 ensures that there exists a unique maximal solution through each initial condition $(t_0,x_0) \in \mathbb{R}^2$.

3.1.2 Autonomous problems.

Corollary 3.4 *Let $D \subset \mathbb{R}^n$ be an open set and $f : D \to \mathbb{R}^n$ a continuous vector field. If $x_0 \in D$ and one of the two following conditions holds*

i) f is locally Lipschitz continuous, *ii) there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $f_{i_0}(x_0) \neq 0$ and f is locally Lipschitz continuous when fixing component i_0,*

then the problem

$$x' = f(x), \quad x(0) = x_0,$$

has a unique local solution.

Example 3.3 *The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$f(x_1, x_2) = (\sin(x_1 x_2), \sqrt{|x_2|} + 1)$$

is not Lipschitz continuous in any neighbourhood of $(0, 0)$. However, $f_2(0, 0) = 1 \neq 0$ and f is locally Lipschitz continuous fixed $i_0 = 2$. Then, by part ii) of corollary 3.4 we have that the problem

$$\begin{cases} x_1' = \sin(x_1 x_2) & , \quad x_1(0) = 0, \\ \\ x_2' = \sqrt{|x_2|} + 1 & , \quad x_2(0) = 0, \end{cases}$$

has a unique local solution.

We have proved in corollary 3.4 that if $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $x_0 \in D$ is not a critical point, then it's enough that f be locally Lipschitz continuous with respect to $n-1$ variables to ensure uniqueness of local solution. In the following example we show that if $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous with respect to $n-2$ variables uniqueness may fail.

Example 3.4 *We consider the problem*

$$\begin{cases} x_1' = 1 & , \quad x_1(0) = 0, \\ x_2' = 1 & , \quad x_2(0) = 0, \\ \vdots & \qquad \vdots \\ x_{n-1}' = 1 & , \quad x_{n-1}(0) = 0, \\ x_n' = \sqrt{|x_1 - x_n|} + 1 & , \quad x_n(0) = 0, \end{cases} \tag{3.6}$$

with $n > 2$. The corresponding vector field is continuous, Lipschitz continuous with respect to $(x_2, x_3, \ldots, x_{n-1})$ and it has no critical points.

However, $x(t)=(t,t,\ldots,t,t)$ *for all* $t\in\mathbb{R}$ *and*

$$\bar{x}(t)=\begin{cases}(t,t,\ldots,t,t) & ,\textit{if}\quad t\le 0,\\ \\ (t,t,\ldots,t,\frac{1}{4}t^2+t) & ,\textit{if}\quad t>0,\end{cases}$$

are two different solutions of problem (3.6).

3.1.3 Scalar problems.

The following result was proved in [3].

Corollary 3.5 *Let* $U\subset\mathbb{R}^2$ *be an open set,* $(t_0,x_0)\in U$ *and* $f:U\subset\mathbb{R}^2\to\mathbb{R}$ *a continuous function. Then there exists a unique local solution of problem (2.3) provided one of the two following conditions holds:*
i) f *is locally Lipschitz continuous with respect to* x*,*
ii) $f(t_0,x_0)\neq 0$ *and* f *is locally Lipschitz continuous with respect to* t*.*

Example 3.5 *We consider the problem*

$$x'=f(t,x)=\mathrm{e}^t+x^{\frac{1}{3}},\quad x(0)=0.$$

It is easy to see that f *is not Lipschitz continuous in any neighbourhood of* $(0,0)$*. However, as* $f(0,0)=1\neq 0$ *and* f *is locally Lipschitz continuous with respect to* t*, condition ii) of corollary (3.5) ensures the existence of a unique local solution.*

In the autonomous case it is obvious that f is locally Lipschitz continuous with respect to t. Therefore, from corollary 3.5 it follows immediately the following result.

Corollary 3.6 *If* $D\subset\mathbb{R}$ *is an open interval,* $f:D\subset\mathbb{R}\to\mathbb{R}$ *is continuous and* $f(x_0)\neq 0$*, then there exists a unique local solution for problem*

$$x'=f(x),\quad x(0)=x_0. \tag{3.7}$$

This result is well-known and was proved by Peano in [9]. It also can be found in [1], theorem 1.2.7. Moreover, problem (3.7), with f not necessarily continuous, has been solved by Binding in [2], where he characterizes the existence and uniqueness of absolutely continuous solutions.

4 An uniqueness result for a class of autonomous planar systems

Let $D \subset \mathbb{R}^2$ be an open set, $P, Q : D \subset \mathbb{R}^2 \to \mathbb{R}$ two continuous functions and $(x_0^1, x_0^2) \in D$. We consider the initial value problem

$$\begin{cases} x_1' = P(x_1, x_2), & x_1(0) = x_0^1, \\ \\ x_2' = Q(x_1, x_2), & x_2(0) = x_0^2. \end{cases} \tag{4.8}$$

The main result of this section is the following.

Theorem 4.1 *Assume that there exists a continuous and strictly positive function $\mu : D \subset \mathbb{R}^2 \to \mathbb{R}$ and a $\mathcal{C}^1$-function $H : D \subset \mathbb{R}^2 \to \mathbb{R}$ such that for all $(x_1, x_2) \in D$ we have that*

$$-\frac{\partial H(x_1, x_2)}{\partial x_2} = \mu(x_1, x_2) P(x_1, x_2) \quad \text{and} \quad \frac{\partial H(x_1, x_2)}{\partial x_1} = \mu(x_1, x_2) Q(x_1, x_2),$$

i.e., the system

$$\begin{cases} x_1' = \mu(x_1, x_2) P(x_1, x_2), \\ x_2' = \mu(x_1, x_2) Q(x_1, x_2), \end{cases}$$

is Hamiltonian.

Then there exists $\alpha > 0$ such that the problem (4.8) has a unique solution in $[-\alpha, \alpha]$, provided that the initial value (x_0^1, x_0^2) is not a critical point, i.e., $(P(x_0^1, x_0^2), Q(x_0^1, x_0^2)) \neq (0, 0)$.

Proof. We suppose without loss of generality that $Q(x_0^1, x_0^2) \neq 0$. The corresponding problem (2.4) associated to problem (4.8) is

$$\begin{cases} y_1' = \frac{P(y_1, r)}{Q(y_1, r)} \quad , & y_1(x_0^2) = x_0^1, \\ \\ y_2' = \frac{1}{Q(y_1, r)} \quad , & y_2(x_0^2) = 0. \end{cases} \tag{4.9}$$

The first equation of this system is a scalar differential equation for which μ is an integrating factor, i.e., the equation

$$y_1' = \frac{\mu(y_1, r) P(y_1, r)}{\mu(y_1, r) Q(y_1, r)} = \frac{-\frac{\partial H(y_1, r)}{\partial r}}{\frac{\partial H(y_1, r)}{\partial y_1}}, \tag{4.10}$$

is an exact equation. Since $\frac{\partial H(x_0^1,x_0^2)}{\partial y_1} \neq 0$ it is well-known that the implicit function theorem implies the existence of $\tilde{\alpha} > 0$ and an unique solution y_1 of (4.10) in $[x_0^2 - \tilde{\alpha}, x_0^2 + \tilde{\alpha}]$ such that $y_1(x_0^2) = x_0^1$. Then, integrating the second equation of (4.9) we deduce that problem (4.9) has a unique solution in $[x_0^2 - \tilde{\alpha}, x_0^2 + \tilde{\alpha}]$ and therefore the existence of $\alpha > 0$ such that (4.8) has a unique solution in $[-\alpha, \alpha]$ follows from theorem 2.2. $\square$

Taking $\mu \equiv 1$ theorem 4.1 applies to Hamiltonian systems. The following corollary of theorem 4.1 was proved by Rebelo in [10] using a different argument.

Corollary 4.2 *Let $D \subset \mathbb{R}^2$ be an open set, $H : D \to \mathbb{R}$ a $\mathcal{C}^1$-function and $(x_0^1, x_0^2) \in D$.*

Then there exists $\alpha > 0$ such that the Hamiltonian system

$$\begin{cases} x_1' = -\frac{\partial H(x_1,x_2)}{\partial x_2} \quad , \quad x_1(0) = x_0^1, \\ \\ x_2' = \frac{\partial H(x_1,x_2)}{\partial x_1} \quad , \quad x_2(0) = x_0^2, \end{cases} \tag{4.11}$$

has a unique solution in $[-\alpha, \alpha]$, provided that (x_0^1, x_0^2) is not a critical point, i.e., $\nabla H(x_0^1, x_0^2) \neq 0$.

Uniqueness of solution may fail when the initial condition is a critical point, as we illustrate in the following example (for a class of examples with this property see remark 1 in [10]).

Example 4.1 *We consider $H : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$H(x_1, x_2) = \int_0^{x_1} \sqrt{|s|}ds - \int_0^{x_2} \sqrt{|s|}ds \quad \textit{for all } (x_1, x_2) \in \mathbb{R}^2,$$

which is $\mathcal{C}^1$. The associated Hamiltonian system is

$$\begin{cases} x_1' = \sqrt{|x_2|}, \\ \\ x_2' = \sqrt{|x_1|}, \end{cases}$$

and $(0,0)$ is a critical point. We have that $x_1(t) = x_2(t) = 0$ for all $t \in \mathbb{R}$ and

$$\bar{x}_1(t) = \bar{x}_2(t) = \begin{cases} 0 & , \textit{if} \quad t \leq 0, \\ \frac{t^2}{4} & , \textit{if} \quad t > 0, \end{cases}$$

are two different solutions through the initial condition $(0,0)$ *(in fact infinitely many solutions exist).*

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