

Existence of a non-zero fixed point for
nondecreasing operators via Krasnoselskii's fixed
point theorem*

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Abstract: In this paper we use Krasnoselskii's fixed point theorem on cone expansions to prove a new fixed point theorem for nondecreasing operators on ordered Banach spaces. Moreover we apply this abstract result to prove the existence of a positive periodic solution for a nonlinear boundary value problem.

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1 Introduction and preliminaries

In [9] the author gives sufficient conditions for the existence of a non-zero fixed point for a nondecreasing mapping $f : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$ by using the properties of the topological degree. This result has applications in the study of economic models and, surprisingly enough, it results equivalent to Brouwer fixed point theorem through the use of the Knaster-Kuratowski-Mazurkiewicz lemma (see [8])

In this paper we use a well-known tool of Nonlinear Analysis, namely Krasnoselskii's fixed point theorem on cone expansions, to extend in section 2 the result of [9] to infinite dimensional Banach spaces. Moreover we apply this abstract result in section 3 to prove the existence of a periodic positive solution for a second order differential equation.

In the sequel we need the following definitions: a subset K of a real Banach space N is a *cone* if it is closed and moreover

- (i) $K + K \subset K$;
- (ii) $\lambda K \subset K$ for all $\lambda \geq 0$;
- (iii) $K \cap (-K) = \{\theta\}$.

A cone K defines the partial ordering in N given by $x \preceq y$ if and only if $y - x \in K$. We use the notation $x \prec y$ for $y - x \in K \setminus \{\theta\}$ and $x \not\preceq y$ for $y - x \notin K$. Note that trough the paper we reserve the symbol " \leq ", and its obvious variants, for the usual order on the real line.

The cone K is *normal* if there exists $c > 0$ such that $\|x\| \leq c\|y\|$ for all $x, y \in N$ with $x \preceq y$. Whenever $int(K) \neq \emptyset$ the symbol $x \ll y$ means $y - x \in int(K)$.

We also need the following fixed point theorem on cone expansions due to Krasnoselskii (see [11, Theorem 13.D]).

THEOREM 1.1 *Let N be a real Banach space with order cone K . Suppose that the operator $T : K \rightarrow K$ is completely continuous and a cone expansion, i.e., there exist $0 < r < R$ such that*

$$Tx \not\leq x \quad \text{for all } x \in K \text{ with } \|x\| = r \quad (1.1)$$

and

$$Tx \not\leq x \quad \text{for all } x \in K \text{ with } \|x\| = R. \quad (1.2)$$

Then T has a fixed point x on K with $r < \|x\| < R$.

Throughout the paper we shall use the following notation: if $b > 0$ and $1 \leq p \leq \infty$ then $L^p(0, b)$ denotes the usual Lebesgue space, $AC[0, b]$ is the set of absolutely continuous functions on $[0, b]$ and $W^{2,1} = \{x \in C^1[0, b] : x' \in AC[0, b]\}$.

2 A positive fixed point theorem

In the following result we present sufficient conditions for a nondecreasing operator defined on an ordered Banach space to have at least a positive non-zero fixed point.

THEOREM 2.1 *Let N be a real Banach space, K a normal cone with nonempty interior and $T : K \rightarrow K$ a nondecreasing and completely continuous operator. Define $S = \{x \in K : Tx \preceq x\}$ and suppose that*

(i) *There exists $\bar{x} \in S$ such that $\bar{x} \gg \theta$.*

(ii) *S is bounded.*

Then there exists $x \in K$, $x \neq \theta$, such that $x = Tx$.

Proof. Since $\bar{x} \in \text{int}(K)$ there exists $r > 0$ such that $\overline{B(\bar{x}, r)} \subset K$. Now, if $x \in K$ with $\|x\| = r$ it is clear that $\bar{x} - x \in \overline{B(\bar{x}, r)} \subset K$ and therefore $x \preceq \bar{x}$. Now we suppose two cases.

Case (I).- Firstly, suppose that there exists $x \in K$ with $\|x\| = r$ such that $Tx \succ x$. Let us define the sequence $x_0 = x$, $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since $x \preceq \bar{x}$, $\bar{x} \in S$ and T is nondecreasing we have that

$$\theta \prec x \preceq x_n = T^n x \preceq T^n \bar{x} \preceq \bar{x} \quad \text{for all } n \in \mathbb{N}.$$

Then the normality of K implies that $\|x_n\| \leq c\|\bar{x}\|$, that is, $\{x_n\}_{n=0}^{\infty}$ is bounded. Now, as T is a completely continuous operator, the sequence $\{Tx_n\}_{n=0}^{\infty} = \{x_n\}_{n=1}^{\infty}$ is relatively compact and therefore there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging to a point x^* . Notice that, since T is nondecreasing, $x_{n_k} \preceq x^*$ for each k . Thus for each $n \geq n_k$ we have $x_{n_k} \preceq x_n \preceq x^*$ and, from the normality of K , it follows that

$$\|x^* - x_n\| \leq c\|x^* - x_{n_k}\|,$$

which implies that the whole sequence $\{x_n\}_{n=0}^{\infty} \rightarrow x^*$. Since T is continuous we deduce that $x^* = Tx^*$ and therefore x^* is a fixed point of T such that $\theta \prec x \preceq x^* \preceq \bar{x}$. In particular $x^* \in K \setminus \{\theta\}$.

Case (II).- To the contrary suppose that $Tx \not\prec x$ for all $x \in K$ with $\|x\| = r$. Now, since S is bounded there exists $R > r$ such that $Tx \not\prec x$ for all $x \in K$ with $\|x\| = R$. Thus Theorem 1.1 implies the existence of a non-zero fixed point also in this case. □

Remark 2.1 (I) Clearly condition (ii) can be replaced by the weaker one

$$(ii)_* \quad \text{There exists } R > 0 \text{ such that } S \cap \{x \in K : \|x\| = R\} = \emptyset$$

(II) Theorem 2.1 combines the monotone iterative technique with the expansion fixed point theorem of Krasnosel'skii. Of course, the classical monotone method

(see [1]) is also applicable under our assumptions but it does not exclude the zero fixed point. Actually, if $T\theta \succ \theta$ and $S \neq \emptyset$ the existence of a non-zero fixed point for the nondecreasing operator T follows by the monotone method under much weaker assumptions than continuity and compactness (see the monograph [5]). Hence, the significant case for us is whenever the zero fixed point is already known, that is, when $T\theta = \theta$. On the other hand, the main difference of Theorem 2.1 with Krasnoselskii's type fixed point theorems (see [4, 11] or the recent generalization [12]) is that we assume condition (1.2) but condition (1.1) is replaced by the monotonicity of the operator. In applications conditions (1.1) and (1.2) traduce on the asymptotic behavior of the nonlinearity on 0 and $+\infty$, respectively. Theorem 2.1 shall allow us to impose only an asymptotic behavior on $+\infty$ but none on 0 .

(III) Under the assumptions of Theorem 2.1 the monotonicity condition of $T : K \rightarrow K$ can be improved by the following one: there exists a real constant $M \geq 0$ such that the operator $Tx + Mx$ is nondecreasing on K . The proof of this fact relies in the equivalence between the fixed point equation $Tx = x$ and $Ax = x$, where

$$Ax \equiv \frac{1}{1+M}(T + M \text{Id})x.$$

Moreover, it is easy to show that $A : K \rightarrow K$ is a condensing operator (see [11, Example 11.7]) and $Ax \preceq x$ if and only if $Tx \preceq x$. (Notice that the combination of monotone iterative technique and Theorem 1.1 given in the proof of Theorem 2.1 are also valid for condensing maps.)

As a particular case of Theorem 2.1 with $N = \mathbb{R}^m$ and the cone $K = \mathbb{R}_+^m$ we obtain the following result.

COROLLARY 2.1 ([9, theorem 5]) *Assume that $f : \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m$ is continuous and nondecreasing. Let $S = \{x \in \mathbb{R}_+^m : f(x) \preceq x\}$. If S is bounded, and if there is an $x' \gg \theta$, $x' \in S$, then there is $x \succeq \theta$, $x \neq \theta$, such that $x = f(x)$.*

3 A periodic boundary value problem

Consider the following periodic problem

$$x''(t) + a(t)x(t) = f(t, x(t)) \text{ a.a. } t \in I = [0, b], \quad x(0) = x(b), \quad x'(0) = x'(b), \quad (3.3)$$

where $a(t) \in L^p(0, b)$, $1 \leq p \leq \infty$ and $f : [0, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a L^1 -Carathéodory function, that is,

- (i) for a.a. $x \in \mathbb{R}$, $f(\cdot, x)$ is measurable;
- (ii) for all $t \in I$, $f(t, \cdot)$ is continuous;
- (iii) for each $r > 0$ there exists $h_r(t) \in L^1(0, b)$ such that

$$|f(t, x)| \leq h_r(t) \quad \text{for a.a. } t \in [0, b] \text{ and all } x \in [-r, r].$$

Moreover we assume the following condition:

- (a0) The Hill's equation $x''(t) + a(t)x(t) = 0$ is nonresonant (i.e., its unique periodic solution is the trivial one) and the corresponding Green's function satisfies that $G(t, s) > 0$ for all $(t, s) \in [0, b] \times [0, b]$.

Whenever $a(t) \equiv k^2$ condition (a0) is equivalent to $0 < k^2 < \left(\frac{\pi}{b}\right)^2$ (see [2]). For a nonconstant function $a(t)$ there exists a L^p -criterion proved by Torres in [10]. For the sake of completeness let us recall such result: consider the Hilbert space

$$H_0^1(0, b) = \{x \in AC[0, b] : x' \in L^2(0, b) \text{ and } x(0) = x(b) = 0\},$$

and define $K(\alpha)$ as the best Sobolev constant in the inequality

$$C\|u\|_\alpha^2 \leq \|u'\|_2^2 \quad \text{for all } u \in H_0^1(0, b),$$

given explicitly by

$$K(\alpha) = \begin{cases} \frac{2\pi}{\alpha b^{1+2/\alpha}} \left(\frac{2}{2+\alpha}\right)^{1-2/\alpha} \left(\frac{\Gamma(1/\alpha)}{\Gamma(1/2+1/\alpha)}\right)^2, & \text{if } 1 \leq \alpha < \infty, \\ \frac{4}{b}, & \text{if } \alpha = \infty, \end{cases}$$

where Γ is the gamma function.

For $a \in L^1(0, b)$ the symbol $a \succ 0$ means that $a(t) \geq 0$ for a. e. $t \in (0, b)$ and $a(t) > 0$ on a set of positive measure and p^* denotes the conjugate of $p \in [1, \infty]$ (that is, $\frac{1}{p} + \frac{1}{p^*} = 1$). Now [10, Corollary 2.3] reads as follows.

THEOREM 3.1 *Assume that $a \in L^p(0, b)$ for some $1 \leq p \leq \infty$, $a \succ 0$ and moreover*

$$\|a\|_p < K(2p^*).$$

Then condition (a0) holds.

It is well known that if the equation $x''(t) + a(t)x(t) = 0$ is nonresonant then a solution of problem (3.3) is equivalent to a fixed point of operator $T : C(I) \rightarrow C(I)$ given by

$$Tx(t) = \int_0^b G(t, s)f(s, x(s))ds \text{ for all } t \in I,$$

where $G(t, s)$ is the corresponding Green's function.

When $a(t) \equiv k^2$ the expression of the Green's function is given by the formula (see [2]):

$$G(t, s) = \frac{1}{2k(1 - \cos kb)} \begin{cases} \sin k(b - t + s) + \sin k(t - s), & \text{if } s \leq t, \\ \sin k(b + t - s) - \sin k(t - s), & \text{if } t \leq s. \end{cases}$$

Assuming (a0), we define

$$m = \min_{t, s \in I} G(t, s) \quad \text{and} \quad M = \max_{t, s \in I} G(t, s).$$

Clearly $0 < m < M$. In particular if $a(t) \equiv k^2$ and $0 < k^2 < \left(\frac{\pi}{b}\right)^2$ it is not difficult to verify that

$$m = \frac{1}{2k} \cot\left(\frac{kb}{2}\right) \quad \text{and} \quad M = \frac{1}{2k} \csc\left(\frac{kb}{2}\right).$$

Now, for each $0 < \gamma < \frac{m}{M} < 1$ consider the cone in $C(I)$

$$K = \{x \in C(I) : x(t) \geq \gamma\|x\|_\infty \text{ for all } t \in I\},$$

which is normal with $c = 1$ and has nonempty interior. Let “ \preceq ” be the order induced in $C(I)$ by the cone K , i.e.,

$$x \preceq y \iff \min_{t \in I} (y(t) - x(t)) \geq \gamma \|y - x\|_\infty.$$

THEOREM 3.2 *Suppose that (a0) and the following assumptions hold:*

(f0) $f(t, x) \geq 0$ for a.a. $t \in I$ and all $x \geq 0$.

(f1) $f(t, \cdot)$ is nondecreasing for a.a. $t \in I$.

(f2) $\lim_{x \rightarrow +\infty} \frac{f(t, x)}{x} = +\infty$ uniformly in t .

(f3) There exists $\bar{x} \in W^{2,1}(I)$ with $\min_{t \in I} \bar{x}(t) > 0$, $\bar{x}(0) = \bar{x}(b)$, $\bar{x}'(0) = \bar{x}'(b)$ and moreover

$$\bar{x}''(t) + a(t)\bar{x}(t) \geq f(t, \bar{x}(t)) \quad \text{for a.a. } t \in I.$$

Then problem (3.3) has a positive solution.

Proof. *Claim 1.* $T(K) \subset K$.

By (f0) we compute

$$\begin{aligned} \min_{t \in I} Tx(t) &= \min_{t \in I} \int_0^b G(t, s) f(s, x(s)) ds \geq \int_0^b m f(s, x(s)) ds \\ &\geq \int_0^b \gamma M f(s, x(s)) ds \geq \gamma \max_{t \in I} \int_0^b G(t, s) f(s, x(s)) ds = \gamma \|Tx\|_\infty, \end{aligned}$$

and then $Tx \in K$ for all $x \in K$.

Claim 2. $T : K \rightarrow K$ is nondecreasing.

From (f1) and similar computations to those of *Claim 1* it follows that if $x \preceq y$ then

$$\min_{t \in I} (Ty(t) - Tx(t)) \geq \gamma \|Ty - Tx\|_\infty.$$

Thus T is nondecreasing with respect to the partial ordering induced by K .

Claim 3. $T : K \rightarrow K$ is completely continuous.

The claim follows by standard arguments.

Claim 4. $\bar{x} \in S = \{x \in K : Tx \preceq x\}$ and $\bar{x} \gg \theta$.

Since $\min_{t \in I} \bar{x}(t) > 0$ we can choose $0 < \gamma < \frac{m}{M} < 1$ small enough such that

$$\min_{t \in I} \bar{x}(t) > \gamma \|\bar{x}\|_\infty,$$

and then $\bar{x} \in \text{int}(K)$.

On the other hand, condition (f3) ensures us the existence of a nonnegative function $h \in L^1(I)$ such that

$$\bar{x}''(t) + a(t)\bar{x}(t) = f(t, \bar{x}(t)) + h(t) \quad \text{for a.e. } t \in I,$$

which is equivalent to

$$\bar{x}(t) - T\bar{x}(t) = \int_0^b G(t, s)h(s)ds.$$

Now, by similar computations to those of *Claim 1* we arrive at

$$\min_{t \in I} (\bar{x}(t) - T\bar{x}(t)) \geq \gamma \|\bar{x} - T\bar{x}\|_\infty,$$

which implies $\bar{x} \preceq T\bar{x}$.

Claim 5. S is bounded.

By (f2) there exists $\alpha > 0$ such that

$$f(t, x) > \frac{x}{\gamma m b} \quad \text{for all } x > \alpha \text{ and all } t \in I.$$

Let $x \in K$ be such that $\min_{t \in I} x(t) > \alpha$. Then

$$f(t, x(t)) > \frac{x(t)}{\gamma m b} \quad \text{for all } t \in I,$$

and we obtain for all $t \in I$

$$\begin{aligned} Tx(t) &= \int_0^b G(t, s)f(s, x(s))ds > \int_0^b G(t, s)\frac{x(s)}{\gamma m b}ds \geq \int_0^b m\frac{\gamma\|x\|_\infty}{\gamma m b}ds \\ &= \|x\|_\infty \geq x(t), \end{aligned}$$

so $x \not\preceq Tx$ and in consequence $x \notin S$. Then, whenever $x \in S$ we have $\min_{t \in I} x(t) \leq \alpha$ and from

$$\alpha \geq \min_{t \in I} x(t) \geq \gamma \|x\|_\infty,$$

it follows that $S \subset \overline{B(\theta, \frac{\alpha}{\gamma})}$.

Finally, from the above claims and Theorem 2.1 it follows the existence of a non trivial fixed point in K for operator T , which is a positive solution of (3.3).

□

Remark 3.1 (1).- Several results on the existence and multiplicity of positive solutions for different kinds of boundary value problems can be found in [6, 7, 10] and references therein.

(2).- In condition (f3) we assume the existence of a positive lower solution $\alpha = \bar{x}$ for equation (3.3). It is well-known that when an anti-maximum principle holds, as in the case studied here, the existence of a lower and an upper solution satisfying the reversed order, $\beta(t) \leq \alpha(t)$ on I , implies the existence of a solution between them (see [2, 3]). Of course, in our case we could choose $\beta \equiv 0$ as an upper solution but then the trivial solution is not excluded. In Theorem 3.2 the a-priori bound on the lower solution set S is the fundamental key to ensure the existence of a positive solution.

References

- [1] Amann, H., Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), no. 4, 620-709.
- [2] Cabada, A., The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems, *J. Math. Anal. Appl.* **185** (1994), 2, 302-320.
- [3] De Coster, C. and Habets, P., *Two-point boundary value problems: lower and upper solutions*, Mathematics in Science and Engineering, 205, Elsevier B. V., Amsterdam, 2006.
- [4] Deimling, K., *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.

- [5] Heikkilä, S. and Lakshmikantham, V., *Monotone iterative techniques for discontinuous nonlinear differential equations*, Marcel Dekker, New York, 1994.
- [6] L. Kong and D. Jiang, Multiple positive solutions of a nonlinear fourth order periodic boundary value problem, *Ann. Polon. Math.*, **69** (1998), no.3, 265-270.
- [7] R. Ma, Multiplicity of positive solutions for second order three-point boundary value problems, *Comput. Math. App.*, **40** (2000), 193-204.
- [8] Persson, H., A fixed point theorem for monotone functions equivalent to Brouwer theorem, Working Paper Series, University of Örebro, no. 1, (2005). (Available at <http://www.oru.se/esi/wps>)
- [9] Persson, H., A fixed point theorem for monotone functions, *Appl. Math. Lett.*, **19**, (2006), 1207–1209.
- [10] Torres, P. J., Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem, *J. Differential Equations*, **190** no. 2 (2003), 643-662.
- [11] Zeidler, E., *Nonlinear functional analysis and its applications. I. Fixed-point theorems*, Springer-Verlag, New York (1986).
- [12] Zhang, G. and Sun, J., A generalization of the cone expansion and compression fixed point theorem and applications, *Nonlinear Anal.* **67** (2007), no. 2, 579–586.