

Positivity and lower and upper solutions for fourth order boundary value problems*

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*First and second authors, partially supported by project HP 2003 – 0080, Spain and Portugal, and by D. G. I. and F.E.D.E.R. project BFM2001-3884-C02-01, and by Xunta of Galicia and F.E.D.E.R. project PGIDT05PXIC20702PN, Spain.

[†]With partial support from the FCT (Fundação para a Ciência e Tecnologia), projects POCTI -ISFL-1-209

Abstract

This paper is devoted to the study the boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)) & \text{for all } t \in I = [0, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

We prove the existence of at least one, two or three solutions in the presence of a pair of, not necessarily ordered, lower and upper solutions.

The proof follows from maximum principles related with the operator $u^{(4)} + Mu$ and Amann's three solutions theorem.

Keywords. Upper and lower solutions, fourth order maximum principles, multiple solutions.

2000 Mathematics Subject Classification. 34B15, 34B27.

1 Introduction

The aim of this paper is to explore the method of lower and upper solutions in order to give some existence and multiplicity results for equations of the form

$$u^{(4)}(t) = f(t, u(t)) \quad \text{for all } t \in I := [0, 1], \quad (1.1)$$

with the boundary conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0. \quad (1.2)$$

Such boundary value problems appear, as it is well known [12], in the theory of hinged beams.

The method of lower and upper solutions is a powerful tool used in Nonlinear Analysis to prove the existence, localization and approximation of solutions for a great variety of boundary value problems.

Roughly speaking, for some kind of second order boundary value problems it is well-known that the existence of a lower solution, α , and an upper solution, β , which are well ordered, that is, $\alpha \leq \beta$, implies the existence of a solution

between them (see [6]). In the recent papers [7, 8, 9] it is shown that also in the presence of nonordered, $\alpha \not\leq \beta$, lower and upper solutions it is possible to obtain the existence of a solution in the set

$$\mathcal{S} = \{u \in \mathcal{C}(I) : \exists t_1, t_2 \in I, u(t_1) \geq \beta(t_1), \alpha(t_2) \geq u(t_2)\}.$$

However the use of lower and upper solutions in boundary value problems of the fourth order, even for the simple boundary conditions (1.2), is heavily dependent of the positiveness properties for the corresponding linear operators. Therefore we found it useful to investigate maximum principles for the operator

$$L_M u := u^{(4)} + M u \tag{1.3}$$

with the boundary conditions (1.2) and to apply them in a systematic way to obtain existence theorems in presence of lower and upper solutions allowing the case where they are not ordered.

The boundary value problem $L_M u = h$ with (1.2) describes the bending of a beam which is attached to an elastic support; u denotes the deviation of the beam under the continuous load function h , and M is an elasticity constant (see [12]).

Maximum principles for operators $u^{(4)}$ and $u^{(4)} + g(x)u''' + h(x)u''$ with the boundary conditions

$$u'(0) \geq 0, \quad u'(1) \leq 0,$$

were given in [5, 10], respectively. On the other hand operator (1.3) with periodic boundary conditions was studied in [4]. However less attention seems to be paid to problem (1.3)-(1.2). In [12] the values of M for which (1.3)-(1.2) is inverse-positive are characterized. We do a more detailed analysis showing also a range of values of M for which (1.3)-(1.2) is inverse-negative and we also study (1.3) with the non homogeneous boundary conditions

$$u(0) = a, \quad u(1) = b, \quad u''(0) = c, \quad u''(1) = d,$$

which shall allow us to consider a wider set of lower and upper solutions. These properties of fourth order linear operators are collected in section 2.

Although these maximum principles seem interesting in themselves, our main purpose is achieved in section 3, where we apply them to prove the existence and multiplicity of solutions for the nonlinear fourth order boundary value problem (1.1) – (1.2), in the presence of nonordered lower and upper solutions.

In our work we shall use the following special case of the “three solutions theorem” due to H. Amman [1, Corollary of Theorem 2].

THEOREM 1.1 *Let X be a closed, bounded, convex subset of a Banach space and let X_1, X_2 be disjoint, closed, convex subsets of X . Let $T : X \rightarrow X$ be a completely continuous and suppose there exist open subsets O_1, O_2 of X with $O_i \subset X_i, i = 1, 2$. Moreover suppose that $T(X_i) \subset X_i$ and that T has no fixed points on $X_i \setminus O_i, i = 1, 2$. Then T has at least three distinct fixed points x, x_1, x_2 with $x_i \in X_i, i = 1, 2$, and $x \in X \setminus (X_1 \cup X_2)$.*

In the sequel we denote by $I = [0, 1]$ and use the following notation: for all $u, v \in \mathcal{C}([a, b])$ given, the symbol $u \not\leq v$ means that there exists $t_0 \in [a, b]$ such that $u(t_0) > v(t_0)$, and the symbol $u \not\geq v$ means that $u(t) \leq v(t)$ for all $t \in [a, b]$ and there exists $t_0 \in [a, b]$ such that $u(t_0) < v(t_0)$.

2 Maximum principles for the operator $u^{(4)} + Mu$

Consider the boundary value problem

$$\begin{cases} u^{(4)}(t) + Mu(t) = h(t), \text{ for all } t \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \quad (2.1)$$

Let $B \subset C^4(I)$ and define the operator $L_M : B \rightarrow \mathcal{C}(I)$ given by

$$[L_M u](t) := u^{(4)}(t) + Mu(t) \text{ for all } t \in I.$$

We say that L_M is **inverse-positive** in B if

$$u \in B, [L_M u](t) \geq 0 \text{ for all } t \in I \implies u(t) \geq 0 \text{ for all } t \in I,$$

and L_M is **strongly inverse-positive** in B if it is inverse positive in B and

$$u \in B, L_M u \not\geq 0 \text{ in } I \implies u(t) > 0 \text{ in } (0, 1).$$

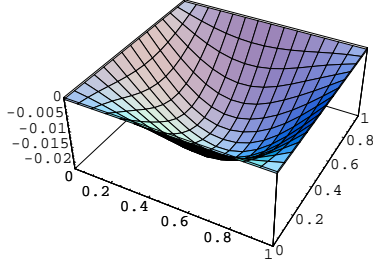


Figure 1: $G_{3.7}(t, s)$

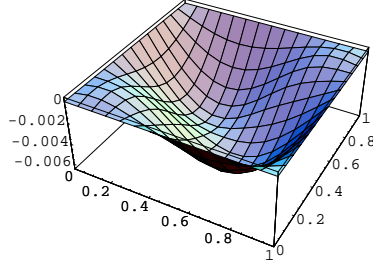


Figure 2: $G_{4.5}(t, s)$

The definitions of **inverse-negative** and **strongly inverse-negative** are similar by reversing the last inequality in the corresponding definition.

In the next results we study the values of $M \in \mathbb{R}$ for which the previous properties hold in suitable domains of Banach spaces.

PROPOSITION 2.1 *Define $W_0 = \{u \in C^4(I) : u(0) = u(1) = u''(0) = u''(1) = 0\}$. The following statements hold:*

- (i) L_M is strongly inverse-positive in $W_0 \iff -\pi^4 < M \leq c_0$;
- (ii) $-\frac{c_0}{4} \leq M < -\pi^4 \implies L_M$ is strongly inverse-negative in W_0 .

Here $c_0 = 4k_0^4 \approx 950.8843$ and $k_0 \approx 3.9266$ is the smallest positive solution of the equation $\tan k = \tanh k$.

Proof. For part (i) see [12], Chapter 2, Section 4.1.3.

To prove (ii) we note that the computation of the Green's function leads to the expression

$$G_m(t, s) = \begin{cases} \frac{\csc(m) \sin(m - mt) \sin(ms) - \operatorname{csch}(m) \sinh(m - mt) \sinh(ms)}{2m^3}, & 0 \leq s \leq t \leq 1, \\ \frac{\csc(m) \sin(m - ms) \sin(mt) - \operatorname{csch}(m) \sinh(m - ms) \sinh(mt)}{2m^3}, & 0 \leq t \leq s \leq 1, \end{cases}$$

where $m = \sqrt[4]{-M}$.

We shall prove that if $\pi < m \leq k_0$ then $G_m(t, s) < 0$ for all $t, s \in (0, 1)$. From the fact that $k_0 < 2\pi$, we have that $\csc(m) < 0$, so, since the Green's

function G_m is symmetric and $\sinh(m) > 0$, we only must show that for all $t, s \in (0, 1)$

$$\sin(mt) \sin(m(1-s)) \sinh(m) - \sin(m) \sinh(m(1-s)) \sin(mt) > 0,$$

which making $\tau = 1 - s$ is equivalent to

$$\frac{\sin(mt) \sin(m\tau)}{\sin(m)} < \frac{\sinh(mt) \sinh(m\tau)}{\sinh(m)} \quad \text{for all } t, \tau \in (0, 1).$$

Clearly it suffices to consider the case $\sin(m\tau) > 0$ and $\sin(mt) < 0$. Since $\sin(x) < \sinh(x)$ for all $x > 0$ it is enough to prove that

$$\frac{\sin(mt)}{\sinh(mt)} > \frac{\sin(m)}{\sinh(m)} \quad \text{for all } t \in (0, 1). \quad (2.2)$$

But this inequality follows immediately from the fact that the derivative of $\frac{\sin(x)}{\sinh(x)}$ is strictly negative in $(0, k_0)$. Therefore since $mt < m \leq k_0$ we have that (2.2) holds. \square

REMARK 2.1 *We remark that in fact it is proven in [12] that for $-\pi^4 < M \leq c_0$ the operator L_M is “strictly” inverse positive, meaning that, $L_M u \geq 0$ in I implies $u(t) > 0$ in $(0, 1)$ and $u'(0) > 0$ and $u'(1) < 0$. This might be seen also studying the sign of partial derivatives of the Green’s function.*

The analogous property holds for case (ii) in the previous proposition, that is, if $\frac{-c_0}{4} \leq M < -\pi^4$ then $L_M u \geq 0$ in I implies $u(t) < 0$ in $(0, 1)$, $u'(0) < 0$ and $u'(1) > 0$. In this case we use (2.2) in the study of the sign of the partial derivatives of the Green’s function.

We shall use these properties in the final theorem of the paper.

REMARK 2.2 *We note that proposition 2.1 has some analogy with the corresponding results for fourth order periodic problems (see theorem 4.1 and lemma 4.1 in [4]).*

In the following theorems we give some comparison principles for operator L_M with a different type of non homogeneous boundary conditions. Before doing that, we state the following lemma.

LEMMA 2.1 *Let h be a continuous function and $a, b, c, d \in \mathbb{R}$ be fixed. Assume that problem*

$$\begin{cases} u^{(4)}(t) + M u(t) = h(t), \text{ for all } t \in I, \\ u(0) = a, u(1) = b, u''(0) = c, u''(1) = d, \end{cases} \quad (2.3)$$

has only the trivial solution for $h \equiv 0$ and $a = b = c = d = 0$. Then (2.3) has a unique solution given by the following expression:

$$u(t) = \int_0^1 G_m(t, s) h(s) ds + a x_m(t) + b x_m(1-t) + c y_m(t) + d y_m(1-t), \quad (2.4)$$

where we denote $M = \pm m^4$ (depending on the sign of M) and x_m and y_m are defined respectively as the unique solutions of the following problems

$$\begin{cases} w^{(4)}(t) + M w(t) = 0, \text{ for all } t \in I, \\ w(0) = 1, w(1) = w''(0) = w''(1) = 0, \end{cases} \quad (2.5)$$

and

$$\begin{cases} w^{(4)}(t) + M w(t) = 0, \text{ for all } t \in I, \\ w(0) = w(1) = 0, w''(0) = 1, w''(1) = 0. \end{cases} \quad (2.6)$$

In the sequel, we shall prove different maximum principles for the case $M \geq 0$.

THEOREM 2.1 *Let $M \geq 0$. Then the linear operator L_M is strongly inverse positive in the space*

$$W_1 = \{u \in \mathcal{C}^4(I) : u(0) \geq 0, u(1) \geq 0, u''(0) = u''(1) = 0\}$$

if and only if $0 \leq M \leq c_1$, where $c_1 = 4k_1^4 \approx 125.137$ and $k_1 \approx 2.365$ is the smallest positive solution of the equation $\tan k = -\tanh k$.

Proof. One can verify, by explicit calculation, that function x_m , defined as the unique solution of (2.5), is given for $m > 0$ by

$$x_m(t) = \frac{-\cos\left(\frac{mt}{\sqrt{2}}\right) \cosh\left(\frac{m(t-2)}{\sqrt{2}}\right) + \cos\left(\frac{m(t-2)}{\sqrt{2}}\right) \cosh\left(\frac{mt}{\sqrt{2}}\right)}{\cos(\sqrt{2}m) - \cosh(\sqrt{2}m)} \quad (2.7)$$

and $x_0(t) = 1 - t$.

Claim. $x_m \geq 0$ in $I \iff m \in [0, \sqrt{2}k_1]$.

First we observe that x_m can not have a double zero in $(0, 1)$, since x_m is the minimizer of the functional

$$\int_0^1 (w''^2(s) + Mw^2(s)) ds$$

in $H^2(0, 1)$ with the boundary conditions $w(0) = 1$ and $w(1) = 0$; if $t_0 \in (0, 1)$ is a double zero of x_m then $x_m(t) = 0$ for all $t \in [t_0, 1]$, which is impossible.

Next we remark that

$$x'_m(1) = \frac{\sqrt{2}m \left(\cosh\left(\frac{m}{\sqrt{2}}\right) \sin\left(\frac{m}{\sqrt{2}}\right) + \cos\left(\frac{m}{\sqrt{2}}\right) \sinh\left(\frac{m}{\sqrt{2}}\right) \right)}{\cos(\sqrt{2}m) - \cosh(\sqrt{2}m)},$$

from which we conclude that $x'_m(1) < 0$ for all $0 < m < \sqrt{2}k_1$ being $\sqrt{2}k_1$ the first positive zero of the equation $x'_m(1) = 0$.

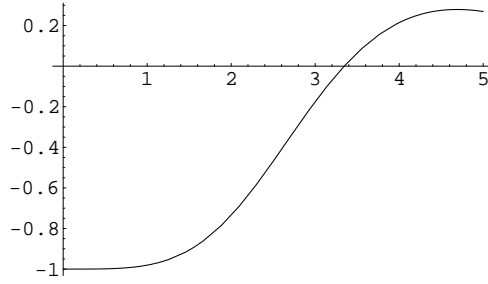


Figure 3: The function $x'_m(1)$

Now suppose that for some $0 < m \leq \sqrt{2}k_1$ the function x_m takes negative values. Using a continuity argument and taking the infimum of such values of $m > 0$ we obtain a $0 < \bar{m} < \sqrt{2}k_1$ such that $x_{\bar{m}}$ has a double zero which is different from 1 since $x'_{\bar{m}}(1) < 0$, but this is a contradiction.

Let $m > \sqrt{2}k_1$ be fixed. We shall prove that x_m has a zero in $(0, 1)$ and, since the zero must be simple, x_m changes sign. In view of (2.7) we have that $x_m(t) = 0$ if and only if $h(t) := f(t) - f(t - 2) = 0$ where

$$f(t) = \frac{\cos\left(\frac{mt}{\sqrt{2}}\right)}{\cosh\left(\frac{mt}{\sqrt{2}}\right)}.$$

It is easy to see that $\frac{\cos s}{\cosh s}$ has its unique absolute maximum at $s = 0$ and an absolute minimum at $s = k_1$. Therefore $h(0) > 0$ and $h(\frac{\sqrt{2}k_1}{m}) \leq 0$. Since $\frac{\sqrt{2}k_1}{m} < 1$ the result follows from Bolzano's theorem.

Now we are in a position to prove the theorem: if L_M is strongly inverse positive in W_1 then clearly x_m must be positive in I and, by the claim, $0 < M \leq c_1$. Conversely if $0 < M \leq c_1$, since $c_1 < c_0$, the conclusion follows by the claim, expression (2.4) and proposition 2.1 (i). \square

Now, we improve the previous result to the particular case of functions u that attain the same non negative value at the endpoints of the interval.

THEOREM 2.2 *Let $M \geq 0$. Then the linear operator L_M is strongly inverse positive in the space*

$$W_2 = \{u \in \mathcal{C}^4(I) : u(0) = u(1) \geq 0, u''(0) = u''(1) = 0\},$$

if and only if $0 \leq M \leq 4\pi^4 \approx 389.636$.

Proof. Since $0 \leq M \equiv m^4 \leq 4\pi^4 < c_0$, it follows from Proposition 2.1, (i) that $G_m > 0$ in $(0, 1) \times (0, 1)$. Thus, by using equation (2.4), it is enough to prove that function

$$w_m(t) = x_m(t) + x_m(1-t)$$

is nonnegative in I if and only if $m \in [0, \sqrt{2}\pi]$.

One can verify, by explicit calculation, that $w_0 \equiv 1$ and, for all $m > 0$, function w_m is given by the following expression:

$$w_m(t) = \left[-\cos\left(\frac{mt}{\sqrt{2}}\right) \cosh\left(\frac{m(t-2)}{\sqrt{2}}\right) + \cos\left(\frac{m(t+1)}{\sqrt{2}}\right) \cosh\left(\frac{m(t-1)}{\sqrt{2}}\right) \right. \\ \left. + \cos\left(\frac{m(t-2)}{\sqrt{2}}\right) \cosh\left(\frac{mt}{\sqrt{2}}\right) - \cos\left(\frac{m(t-1)}{\sqrt{2}}\right) \cosh\left(\frac{m(t+1)}{\sqrt{2}}\right) \right] \\ / [\cos(\sqrt{2}m) - \cosh(\sqrt{2}m)]$$

Claim 1.- If $w_m(t) \geq 0$ for all $t \in I$ then $\min_{t \in I} w_m(t) = w_m(1/2)$.

From the definition, it is obvious that function w_m is symmetric with respect to $t = 1/2$. Moreover if $w_m(t) \geq 0$ then w_m'' is concave and then using the boundary conditions we conclude that w_m is convex. Therefore the claim follows.

Claim 2.- The set $A = \{m \geq 0 : w_m \geq 0 \text{ in } I\}$ is an interval.

Clearly A is nonempty because $0 \in A$. If $m_1 \in [0, \sqrt{2}k_0]$ is such that $m_1 \in A$ and $0 \leq m_2 < m_1$ then $m_2 \in A$. Indeed, from the equations

$$\begin{cases} w_i^{(4)} + m_i^4 w_i = 0, \\ w_i(0) = w_i(1) = 1, \quad w_i''(0) = w_i''(1) = 0, \end{cases}$$

for $i = 1, 2$, it follows that $w = w_2 - w_1$ satisfies the problem

$$\begin{cases} w^{(4)} + m_2^4 w = (m_1^4 - m_2^4)w_1 \not\geq 0, \\ w(0) = w(1) = 0, \quad w''(0) = w''(1) = 0. \end{cases}$$

Hence by Proposition 2.1, (i) we obtain that $w_2 > w_1$ in $(0, 1)$ and, as consequence, $m_2 \in A$.

Since

$$w_m\left(\frac{1}{2}\right) = \frac{2 \cos\left(\frac{m}{2\sqrt{2}}\right) \cosh\left(\frac{m}{2\sqrt{2}}\right)}{\cos\left(\frac{m}{\sqrt{2}}\right) + \cosh\left(\frac{m}{\sqrt{2}}\right)},$$

the preceding argument shows that $m_1 > \sqrt{2}k_0$ and $m_1 \in A$ is impossible: otherwise we obtain that $m_2 \in A$ for all $m_2 \in (\sqrt{2}\pi, \sqrt{2}k_0)$ but $w_{m_2}(1/2) < 0$ for all $m_2 \in (\sqrt{2}\pi, \sqrt{2}k_0)$. Therefore A is an interval contained in $[0, \sqrt{2}k_0]$.

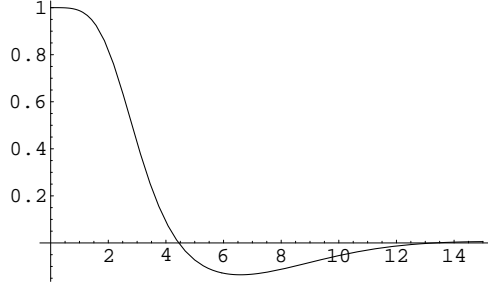


Figure 4: The function $w_m(1/2)$

Claim 3.- $A = [0, \sqrt{2}\pi]$

By continuity A is a closed interval $[0, l]$. If $l < \sqrt{2}\pi$ then $w_l(t) \geq w_l(1/2) > 0$ for all $t \in I$, and again by continuity $w_m \geq 0$ for all m in a small enough right neighborhood of l , a contradiction. On the other hand $l > \sqrt{2}\pi$ is impossible because $w_m(1/2) < 0$ in a right neighborhood of $\sqrt{2}\pi$. \square

To study the case in which the solution of problem (2.3) vanishes at $t = 0$ and $t = 1$ but the second derivatives at such points could be different from zero, we must study the function y_m defined as the unique solution of (2.6).

THEOREM 2.3 *Let $M \geq 0$. Then the linear operator L_M is strongly inverse positive in the space*

$$W_3 = \{u \in C^4(I) : u(0) = u(1) = 0, u''(0) \leq 0, u''(1) \leq 0\}, \quad (2.8)$$

if and only if $0 \leq M \leq c_0$, where $c_0 \approx 950.8843$ is given in Proposition 2.1 (i).

Proof. It is enough to take into account that

$$y_m(t) = \frac{-\sin\left(\frac{mt}{\sqrt{2}}\right) \sinh\left(\frac{m(t-2)}{\sqrt{2}}\right) + \sin\left(\frac{m(t-2)}{\sqrt{2}}\right) \sinh\left(\frac{mt}{\sqrt{2}}\right)}{m^2 (\cos(\sqrt{2}m) - \cosh(\sqrt{2}m))},$$

$$y'_m(1) = \frac{\sqrt{2}e^{\frac{m}{\sqrt{2}}} \left((1 - e^{\sqrt{2}m}) \cos\left(\frac{m}{\sqrt{2}}\right) + (1 + e^{\sqrt{2}m}) \sin\left(\frac{m}{\sqrt{2}}\right) \right)}{m(1 + e^{2\sqrt{2}m} - 2e^{\sqrt{2}m} \cos(\sqrt{2}m))}$$

and use similar arguments to those of the proof of proposition 2.1 (i) and theorem 2.1. \square

REMARK 2.3 *Note that, contrary to Theorem 2.2, in this situation we do not consider the case $u''(0) = u''(1) \geq 0$, because now we have the same estimate as in Proposition 2.1 (i), and so the result cannot be improved.*

As a conclusion of these previous results we arrive at the following corollaries.

COROLLARY 2.1 *Let $M \geq 0$. Then the linear operator L_M is strongly inverse positive in the space*

$$W_4 = \{u \in C^4(I) : u(0) \geq 0, u(1) \geq 0, u''(0) \leq 0, u''(1) \leq 0\}$$

if and only if $0 \leq M \leq c_1$, where c_1 is defined in theorem 2.1.

COROLLARY 2.2 *Let $M \geq 0$. Then the linear operator L_M is strongly inverse positive in the space*

$$W_5 = \{u \in C^4(I) : u(0) = u(1) \geq 0, u''(0) \leq 0, u''(1) \leq 0\}$$

if and only if $0 \leq M \leq 4\pi^4 \approx 389.636$.

For the case $M = -m^4 < 0$ a detailed analysis of the functions x_m , w_m and y_m discloses that x_m and w_m always change sign and $y_m \geq 0$ if and only if $m \in (\pi, k_0]$. We state without proof the conclusion in this case.

THEOREM 2.4 *Let $-\frac{c_0}{4} \leq M < -\pi^4$, where c_0 is given in proposition 2.1.*

Then the linear operator L_M is strongly inverse negative in the space W_3 defined in (2.8).

3 Some applications to nonlinear boundary value problems

In this section we deal with the problem (1.1) – (1.2) with $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ a continuous function.

For the purpose of this paper we shall use the following definitions.

DEFINITION 3.1 *We say that $\alpha \in C^4(I)$ is a lower solution of problem (1.1) – (1.2) if*

$$\begin{aligned} \alpha^{(4)}(t) &\leq f(t, \alpha(t)) \quad \text{for all } t \in I, \\ \alpha(0) &\leq 0, \alpha(1) \leq 0, \quad \alpha''(0) \geq 0, \alpha''(1) \geq 0. \end{aligned}$$

Further, $\alpha \in C^4(I)$ is a strict lower solution if it is a lower solution and, moreover

$$\alpha^{(4)}(t_0) < f(t_0, \alpha(t_0)) \quad \text{for some } t_0 \in I.$$

The concept of an upper (strict upper) solution $\beta \in C^4(I)$ is similar by reversing the above inequalities.

Before proving existence results for problem (1.1) – (1.2), we consider the following inequalities that will appear in the remaining of the paper:

$$(L_1) \quad f(t, \alpha(t)) + M\alpha(t) \leq f(t, u) + Mu \leq f(t, \beta(t)) + M\beta(t), \quad \alpha(t) \leq u \leq \beta(t),$$

$$(L_2) \quad f(t, \alpha(t)) - M\alpha(t) \leq f(t, u) - Mu \leq f(t, \beta(t)) - M\beta(t), \quad \beta(t) \leq u \leq \alpha(t).$$

Proposition 2.1 coupled with Theorems 2.1, 2.2, 2.3 and 2.4 allows us to obtain the following existence result in the presence of ordered lower and upper solutions.

THEOREM 3.1 *Suppose that $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Let α and β be lower and upper solutions, respectively, for problem (1.1) – (1.2). Then we have the following results:*

- (I) *If $\alpha \leq \beta$ and moreover there exists $0 \leq M \leq c_1$ for which property (L_1) holds, then there exists a solution u of (1.1) – (1.2) in $[\alpha, \beta]$.*
- (II) *If $\alpha \leq \beta$, $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$, and moreover there exists $0 \leq M \leq c_0$ for which property (L_1) holds, then there exists a solution u of (1.1) – (1.2) in $[\alpha, \beta]$.*
- (III) *If $\alpha \leq \beta$ with $\alpha(0) = \alpha(1) \leq 0 \leq \beta(0) = \beta(1)$ and moreover there exists $0 \leq M \leq 4\pi^4$ for which property (L_1) holds, then there exists a solution u of (1.1) – (1.2) in $[\alpha, \beta]$.*
- (IV) *If at least one of the conditions (I), (II) or (III) holds and the lower solution α (resp. the upper solution β) is strict and $u \in [\alpha, \beta]$ is any solution of problem (1.1) – (1.2) then $\alpha(t) < u(t)$ for all $t \in (0, 1)$ (resp. $\beta(t) > u(t)$ for all $t \in (0, 1)$).*
- (V) *If $\beta \leq \alpha$, $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = 0$, and moreover there exists $\pi^4 < M \leq \frac{c_0}{4}$ for which property (L_2) holds, then there exists a solution u of (1.1) – (1.2) in $[\beta, \alpha]$.*
Moreover if the lower solution α (resp. the upper solution β) is strict and $u \in [\beta, \alpha]$ is any solution of problem (1.1) – (1.2) then $\alpha(t) > u(t)$ for all $t \in (0, 1)$ (resp. $\beta(t) < u(t)$ for all $t \in (0, 1)$).

Proof. We shall prove only part (I) because the proofs of the other ones are similar.

We define $K_M : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ in the following way: for each $h \in \mathcal{C}(I)$ let $K_M h$ be the unique solution of problem

$$\begin{cases} u^{(4)}(t) + Mu(t) = h(t) & \text{for all } t \in I, \\ u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0. \end{cases}$$

Let $N_M : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ be the Nemytskii operator given for each $h \in \mathcal{C}(I)$ by

$$(N_M h)(t) = f(t, h(t)) + Mh(t) \quad \text{for all } t \in I,$$

and finally we define $T_M : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ as

$$T_M = K_M \circ N_M, \tag{3.1}$$

that is, for each $h \in \mathcal{C}(I)$ the function $T_M h$ is the unique solution of problem

$$\begin{cases} u^{(4)}(t) + Mu(t) = f(t, h(t)) + Mh(t) & \text{for all } t \in I, \\ u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0. \end{cases}$$

Clearly the operator T_M is completely continuous.

We consider in $\mathcal{C}(I)$ the pointwise partial ordering

$$u, v \in \mathcal{C}(I), \quad u \leq v \iff u(t) \leq v(t) \quad \text{for all } t \in I,$$

and for $u \leq v$ we define the functional interval

$$[u, v] = \{w \in \mathcal{C}(I) : u \leq w \leq v\}.$$

Claim 1.- $K_M : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ is nondecreasing.

Let $h_1, h_2 \in \mathcal{C}(I)$ with $h_1 \leq h_2$ and put $u_i = K_M h_i$, $i = 1, 2$. Then $w = u_2 - u_1$ satisfies

$$\begin{cases} w^{(4)}(t) + Mw(t) = h_2(t) - h_1(t) \geq 0 & \text{for all } t \in I, \\ w(0) = w(1) = 0, \quad w''(0) = w''(1) = 0, \end{cases}$$

and since $0 \leq M \leq c_1 < c_0$, from Proposition 2.1, (i) it follows that $w \geq 0$ and hence $u_1 \leq u_2$.

Claim 2.- $\alpha \leq T_M \alpha$ and $T_M \beta \leq \beta$.

Since α is a lower solution we have that

$$(T_M \alpha)^{(4)}(t) + M(T_M \alpha)(t) = f(t, \alpha(t)) + M\alpha(t) \geq \alpha^{(4)}(t) + M\alpha(t) \quad \text{for all } t \in I.$$

Thus $w = T_M\alpha - \alpha$ satisfies that

$$\begin{cases} w^{(4)}(t) + Mw(t) \geq 0 & \text{for all } t \in I, \\ w(0) \geq 0, w(1) \geq 0, w''(0) \leq 0, w''(1) \leq 0, \end{cases}$$

and then by Corollary 2.1 we deduce that $w = T_M\alpha - \alpha \geq 0$.

In an analogous way we can prove that $T_M\beta \leq \beta$.

Claim 3.- $T_M([\alpha, \beta]) \subset [\alpha, \beta]$.

Let $u \in [\alpha, \beta]$. By our hypothesis we have that

$$f(t, \alpha(t)) + M\alpha(t) \leq f(t, u) + Mu \leq f(t, \beta(t)) + M\beta(t), \text{ for all } t \in I,$$

and by *Claim 1* we deduce that

$$T_M\alpha \leq T_Mu \leq T_M\beta.$$

Finally *Claim 2* implies that $T_Mu \in [\alpha, \beta]$.

Conclusion.- The interval $[\alpha, \beta]$ is a closed, convex, bounded and nonempty subset of the Banach space $C(I)$. Then by *Claim 3* we can apply Schauder's fixed point theorem to obtain the existence of a fixed point of T_M , which obviously is a solution of problem (1.1) – (1.2) in $[\alpha, \beta]$.

Note that if condition (IV) holds, from the strongly inverse positive character of operator L_M in W_4 , we deduce from *Claims 1 – 3* that $\alpha < T_Mu$ in $(0, 1)$. \square

REMARK 3.1 1. *It is well know that for a second order differential equation, with periodic, Neumann or Dirichlet boundary conditions, the existence of a well ordered pair of lower and upper solutions $\alpha \leq \beta$ is enough to ensure the existence of a solution in the sector enclosed by them (see [6]).*

However this result it is not true for problem (1.1) – (1.2) as we shall show following some ideas of [3]. Indeed, let $M_1 > c_0$. By Proposition 2.1, (i) we know that L_{M_1} is not inverse positive on W_0 . Then there exists $h \in C(I)$, $h \geq 0$, such that the solution of

$$\begin{cases} u^{(4)}(t) + M_1u(t) = h(t) & \text{for all } t \in I, \\ u(0) = u(1) = 0, u''(0) = u''(1) = 0, \end{cases}$$

is not positive. Moreover we can choose $h(t) \leq 1$ (dividing by $\|h\|_\infty$ if it is necessary).

Clearly, $\alpha(t) = 0 \leq \beta(t) = 1$ are lower and upper solutions for the problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)) \equiv -M_1 u(t) + h(t) & \text{for all } t \in I, \\ u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0, \end{cases}$$

but its unique solution u satisfies $u \not\leq 0 = \alpha$ and therefore there is no solution between α and β .

2. We point out that all the statements of Theorem 3.1 can be established in the framework of L^1 -Carathéodory functions.

3. We also remark that if we replace conditions (L_1) and (L_2) in the previous theorem by the corresponding ones

$$(L_1^*) \quad f(t, u) - f(t, v) \leq M(v - u) \quad \text{for all } t \in I \text{ and all } \alpha(t) \leq u \leq v \leq \beta(t),$$

$$(L_2^*) \quad f(t, v) - f(t, u) \leq M(v - u) \quad \text{for all } t \in I \text{ and all } \beta(t) \leq u \leq v \leq \alpha(t),$$

we can develop the monotone method obtaining a stronger result, namely, the existence of monotone sequences which converge to the extremal solutions between the lower and the upper solution (or between the upper and the lower solution in case (V)).

As a first consequence of the previous theorem we obtain the following result.

PROPOSITION 3.1 *The operator L_M is strongly inverse negative on W_0 if and only if $M \in [-c_0/4, -\pi^4]$.*

Proof. Using Proposition 2.1 (ii), we only must prove the first implication. To this end, define

$$B = \{m > \pi, \text{ such that } L_{-m^4} \text{ is strongly inverse negative in } W_0\}.$$

We will prove that B is an interval.

Let $m_1 < m_2$ belonging to B , assume that $m_3 \in (m_1, m_2)$.

Fix $h \not\geq 0$ a continuous function. From the definition of B , we know that the unique solutions of problems $L_{-m_i^4} u_i = h$, $u_i \in W_0$ satisfy $u_i < 0$ in $(0, 1)$,

$i = 1, 2$. Moreover

$$L_{-m_1^4}(u_1 - u_2) = (m_1^4 - m_2^4)u_2 \geq 0, \quad u_1 - u_2 \in W_0,$$

and, as consequence, $u_1 < u_2$ in $(0, 1)$.

By defining $f(t, u) = m_3^4 u + h(t)$ it is easy to verify that u_1 is an upper solution and u_2 is a lower solution of the corresponding problem. Moreover, function f satisfies condition (L_2) for $M \equiv m_2^4$. So we are in the hypotheses of Theorem 3.1 (V) and, as a consequence, the equation $L_{-m_3^4} u = h$, $u \in W_0$ has a solution $u_3 \in (u_2, u_1)$, in particular $u_3 < 0$ in $(0, 1)$.

In particular we have shown that the operator $L_{-m_3^4}$ restricted to W_0 has range equal to $\mathcal{C}(I)$. By the linear Fredholm alternative the operator $L_{-m_3^4}$ is invertible and then $m_3 \in B$.

Now, by using the expression of G_m (G_m given in Proposition 2.1), we arrive at

$$\frac{d}{dt}G_m(t, t) = \frac{\csc(m) \sin(m(1 - 2t)) - \operatorname{csch}(m) \sinh(m(1 - 2t))}{2m^2}.$$

It is easy to verify, using the fact that $\frac{\sin x}{\sinh x}$ becomes increasing in a right neighborhood of k_0 (cf. the proof of proposition 2.1), that there exist $\epsilon > 0$ such that $\frac{d}{dt}G_m(t, t) > 0$ for all $t \in (0, \epsilon)$ and $m \in (k_0, k_0 + \epsilon)$. So $B = (\pi, \sqrt[4]{c_0/4}]$.
□

As a consequence of theorem 2.4 and proposition 3.1 we deduce the following result.

COROLLARY 3.1 *The operator L_M is strongly inverse negative on W_3 if and only if $M \in [-c_0/4, -\pi^4)$.*

In the following theorem we give existence and multiplicity results in presence of lower and upper solutions. In order to simplify the proof we consider the case where the lower and the upper solutions take equal values at the end points of I . However a similar theorem can be proved in the more general case replacing the constant $4\pi^4$ with c_1 .

THEOREM 3.2 *Suppose that $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that for some $A > 0$ we have*

$$|f(t, x)| \leq A \quad \text{for all } (t, x) \in I \times \mathbb{R},$$

and for some $0 \leq M \leq 4\pi^4$

$$f(t, x) - f(t, y) \leq M(y - x) \quad \text{for all } t \in I \text{ and } x \leq y.$$

Then the following results hold:

- (i) *(A priori bounds and existence of solutions) The problem (1.1) – (1.2) has a solution and moreover there exists $k > 0$ such that any solution u of problem (1.1) – (1.2) satisfies $\|u\|_\infty \leq k$.*
- (ii) *(Localization of solutions)*

1. *If α is a lower solution with $\alpha(0) = \alpha(1)$ then problem (1.1) – (1.2) has a solution u satisfying*

$$\alpha(t) \leq u(t) \quad \text{for all } t \in I.$$

2. *If β is an upper solution with $\beta(0) = \beta(1)$ then problem (1.1) – (1.2) has a solution u satisfying*

$$u(t) \leq \beta(t) \quad \text{for all } t \in I.$$

- (iii) *(Multiplicity of solutions) Let α and β be lower and upper solutions with $\alpha(0) = \alpha(1)$ and $\beta(0) = \beta(1)$ such that $\alpha \not\leq \beta$. Then problem (1.1) – (1.2) has at least two different solutions u_1 and u_2 with*

$$\alpha(t) \leq u_1(t) \quad \text{and} \quad u_2(t) \leq \beta(t) \quad \text{for all } t \in I.$$

If moreover α and β are strict then there exist at least three different solutions u_1, u_2 and u_3 , with

$$\alpha(t) \leq u_1(t) \quad \text{and} \quad u_2(t) \leq \beta(t) \quad \text{for all } t \in I,$$

and $u_3 \in \mathcal{S}$ where

$$\mathcal{S} = \{u \in \mathcal{C}([0, T]) : \exists t_1, t_2 \in [0, T], u(t_1) \geq \beta(t_1), \alpha(t_2) \geq u(t_2)\}. \quad (3.2)$$

Proof. (i).- Define the operator $T : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ given for each $h \in \mathcal{C}(I)$ by

$$Th(t) = \int_0^1 G_0(t, s)f(s, h(s))ds \quad \text{for all } t \in I,$$

where $G_0(t, s)$ is the Green's function of the problem

$$\begin{cases} u^{(4)}(t) = \sigma(t) & \text{for all } t \in I, \\ u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0. \end{cases}$$

Clearly T is completely continuous and the fixed points of T are the solutions of (1.1) – (1.2). On the other hand, since f is bounded there exists $k > 0$ such that

$$\|Th\|_\infty < k \quad \text{for all } h \in \mathcal{C}(I).$$

Then we have the *a priori* bounds on the solutions and moreover Schauder's fixed point theorem yields the existence of a solution of problem (1.1) – (1.2).

(ii).- We only write the proof for the first case because the second case is similar.

Step 1.- The modified problem.

We fix $0 < \varepsilon < \min\{M, \pi^4\}$ and for each $r > 0$ we define

$$f_r(t, u) = \begin{cases} f(t, -r) - \varepsilon(u + r), & \text{if } u < -r, \\ f(t, u), & \text{if } |u| \leq r, \\ f(t, r) - \varepsilon(u - r), & \text{if } u > r, \end{cases}$$

and consider the modified problem

$$\begin{cases} u^{(4)}(t) = f_r(t, u(t)) & \text{for all } t \in I, \\ u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0, \end{cases} \quad (3.3)$$

Step 2.- There exists $d > 0$ such that any solution u of (3.3) is such that $\|u\|_\infty \leq d$, independently of r .

By considering the odd extension of function u to the interval $[-1, 1]$, we can apply the classical Wirtinger – type inequality for functions g defined in a real interval $[a, b]$, that have an absolutely continuous first derivative on (a, b) , its Fourier series is uniformly convergent on (a, b) , $g(a) = g(b)$ and $\int_a^b g(s) ds = 0$ (see inequality (4.2) in [11, Chapter II]). So, we have

$$\pi^4 \int_0^1 u^2(s)ds \leq \int_0^1 u''^2(s)ds. \quad (3.4)$$

On the other hand as f is bounded, there exists a constant $C > 0$ independent of r such that

$$|f_r(t, u)| \leq \varepsilon|u| + C \quad \text{for all } (t, u) \in I \times \mathbb{R},$$

and thus if u is a solution of (3.3), multiplying the equation by u and integrating we have

$$\int_0^1 u''^2(s) ds \leq \varepsilon \int_0^1 u^2(s) ds + C \int_0^1 |u(s)| ds. \quad (3.5)$$

Now Holder's inequality implies that

$$\left(\int_0^1 |u(s)| ds \right)^2 \leq \int_0^1 u^2(s) ds. \quad (3.6)$$

Then, since $0 < \varepsilon < \pi^4$, from (3.4), (3.5) and (3.6) it follows that u and u'' are bounded in $L^2(0, 1)$, independently of r .

Now since $u(0) = u(1)$ there exists $t_0 \in (0, 1)$ such that $u'(t_0) = 0$ and then for all $t \in I$ we have that

$$|u'(t)| = |u'(t) - u'(t_0)| = \left| \int_{t_0}^t u''(s) ds \right| \leq \int_0^1 |u''(s)| ds \leq \left(\int_0^1 u''^2(s) ds \right)^{\frac{1}{2}},$$

which implies that u' is also bounded in $L^\infty(0, 1)$, independently of r .

Finally from [2, Theorem VIII.7 and Proposition VIII.12] it follows that

$$\|u\|_\infty \leq C_1 \|u\|_{W^{1,2}} \leq C_2 \|u'\|_{L^2},$$

and therefore there exists $d > 0$, independently of r , such that

$$\max\{\|u\|_\infty, \|u'\|_\infty\} \leq d.$$

Step 3.- There exists a solution u of (3.3) for all r large enough.

Let $r > \|\alpha\|_\infty$. It is easy to see that

$$\beta_1(t) = \frac{A}{\varepsilon} + r + 1,$$

is a strict upper solution and moreover for all $t \in I$ we have

$$f_r(t, \alpha(t)) + M\alpha(t) \leq f_r(t, u) + Mu \leq f_r(t, \beta_1(t)) + M\beta_1(t) \quad \text{for } \alpha(t) \leq u \leq \beta_1(t).$$

Now, by applying Theorem 3.1, (III), with f_r instead of f the existence of a solution in $[\alpha, \beta_1]$ follows.

Conclusion: By steps 1 and 2 taking $r > d$ we obtain the existence of a solution $u \geq \alpha$ of (3.3) with $\|u\|_\infty \leq d < r$. Hence u is also a solution of the original problem (1.1) – (1.2).

(iii).- The existence of two solutions follows from (ii) and the fact that $\alpha \not\leq \beta$.

Now suppose that α and β are strict. Choose $r > \max\{d, \|\alpha\|_\infty, \|\beta\|_\infty\}$ and define

$$\alpha_1(t) = -\frac{A}{\varepsilon} - r - 1 \quad \text{and} \quad \beta_1(t) = \frac{A}{\varepsilon} + r + 1,$$

which clearly are strict lower and upper solutions, respectively, for the modified problem with f_r .

Let T_M be given by (3.1) with f_r instead of f .

1. If $\alpha(0) = \alpha(1) < 0$ and $\beta(0) = \beta(1) > 0$ we work in the space $\mathcal{C}(I)$ and set

$$X = \{u \in \mathcal{C}(I) : \alpha_1(t) \leq u(t) \leq \beta_1(t) \text{ in } I\}$$

$$X_1 = \{u \in X : \alpha(t) \leq u(t) \leq \beta_1(t) \text{ in } I\},$$

$$X_2 = \{u \in X : \alpha_1(t) \leq u(t) \leq \beta(t) \text{ in } I\},$$

$$O_1 = \{u \in X : \alpha(t) < u(t) < \beta_1(t) \text{ in } I\},$$

$$O_2 = \{u \in X : \alpha_1(t) < u(t) < \beta(t) \text{ in } I\}.$$

2. If (to fix ideas) $\alpha(0) = \alpha(1) = 0$ and $\beta(0) = \beta(1) > 0$ (the remaining cases being treated with the obvious changes), we then work in the space $\mathcal{C}_0^1(I) := \{u \in \mathcal{C}^1(I) : u(0) = u(1) = 0\}$ where we set

$$X = \{u \in \mathcal{C}_0^1(I) : \alpha_1(t) \leq u(t) \leq \beta_1(t) \text{ in } I, \|u'\|_\infty \leq D\},$$

$$X_1 = \{u \in X : \alpha(t) \leq u(t) \leq \beta_1(t) \text{ in } I\},$$

$$X_2 = \{u \in X : \alpha_1(t) \leq u(t) \leq \beta(t) \text{ in } I\},$$

$$O_1 = \{u \in X : \alpha(t) < u(t) < \beta_1(t) \text{ in } (0,1), u'(0) > \alpha'(0), u'(1) < \alpha'(1), \}$$

$$O_2 = \{u \in X : \alpha_1(t) < u(t) < \beta(t) \text{ in } (0,1)\}$$

where $D > \max\{d, \sup_{\alpha_1 < h(t) < \beta_1} \|T_M h\|_{\mathcal{C}^1(I)}\}$.

In any case X_i is a closed, bounded, convex subset of X and O_i is an open set with $O_i \subset X_i$. Moreover $T_M(X_i) \subset X_i$, $i = 1, 2$. We claim that T_M has no fixed point in $X_i \setminus O_i$, $i = 1, 2$. Indeed, if $u \in X_1$ is a fixed point of T_M then the function $z = u - \alpha$ satisfies the inequality $z^{(4)} + Mz \geq 0$ in I and moreover $z(0) = z(1) = z''(0) = z''(1) = 0$. Then by Proposition 2.1, (i) and remark 2.1 we obtain that in any case $u \in O_1$. The same is true for X_2 and O_2 . Therefore Theorem 1.1 implies the existence of three solutions with the desired properties. \square

REMARK 3.2 *We can state a similar theorem if we suppose that for some $\pi^4 < M \leq \frac{c_0}{4}$*

$$f(t, x) - f(t, y) \geq M(x - y) \quad \text{for all } t \in I \text{ and } x \leq y,$$

the lower and the upper solutions satisfy

$$\alpha(0) = \alpha(1) = 0 \quad \text{and} \quad \beta(0) = \beta(1) = 0$$

and moreover in the case (iii) we have $\beta \not\leq \alpha$. The conclusions are similar by reversing the inequalities.

EXAMPLE 3.1 *Consider the function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$f(t, u) = (16\pi^4 + 2\sin(2\pi t))\delta(u) + 2\cos^2(2\pi t) - 2 + g(u),$$

where $\delta : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\delta(u) = \begin{cases} -1, & \text{if } u < -1, \\ u, & \text{if } |u| \leq 1, \\ 1, & \text{if } u > 1, \end{cases}$$

and $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfies that $g(0) = 0$, $0 < g(u) < 1$ for all $u \neq 0$ and for some $M \in [0, 4\pi^4]$ the function $g(u) + Mu$ is increasing in u .

It is easy to check that $\alpha(t) = \sin(2\pi t)$ and $\beta(t) = 0$ for all $t \in I$ are strict lower and upper solutions, respectively, and $\alpha \not\leq \beta$. Moreover f satisfies the conditions of Theorem 3.2, (iii) and therefore it follows the existence of at least three solutions for problem (1.1) – (1.2).

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