On first order ordinary differential equations with non-negative right-hand sides

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Abstract:

Existence of extremal solutions for scalar initial value problems is investigated. We shall concentrate upon nonlinearities having constant sign, which lead to new existence results.

Keywords: Subfunctions, ordinary differential equations, Carathéodory conditions, discontinuous differential equations, singular differential equations.

1 Introduction

C. Carathéodory proved in [6] that the problem

$$x'(t) = f(t, x(t)) \quad \text{for a.a. } t \in [t_0, T],\ x(t_0) = x_0, \tag{1.1}$$

has at least one absolutely continuous solution provided that the right-hand side $f : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$(C1)$ for all $x \in \mathbb{R}^n$, $f(\cdot, x)$ is measurable;

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$(C2)$ for a.a. $t \in [t_0, T]$, $f(t, \cdot)$ is continuous; and

$(C3)$ there exists $M \in L^1(t_0, T)$ such that for a.a. $t \in [t_0, T]$ and all $x \in \mathbb{R}^n$

$$|f(t, x)| \le M(t).$$

It is worthwhile to note that if x is a solution of (1.1) in the Carathéodory sense and, moreover, the composition $f(\cdot, x(\cdot))$ is continuous, then x is a classical $\mathcal{C}^1$ solution.

Carathéodory's result has been generalized once and again, especially in the scalar case. We can quote here some outstanding works such as [1, 2, 3, 4, 7, 8, 9, 10] (see also references therein), where a variety of extra information on the solutions and finer existence conditions can be found. We can mention Goodman's paper, [8], where a greatest and a least solutions are proven to exist by adapting Peano's subfunction approach, [15], to the scalar $(C1) - (C2) - (C3)$ case. One of the essential recent steps in this line of research is due to Biles, [1], who (still in the $\mathbb{R}^n$ case) replaced $(C1)$ and $(C2)$ by

$(B1)$ for all $x \in \mathcal{C}([t_0, T], \mathbb{R}^n)$, the composition $f(\cdot, x(\cdot))$ is measurable;

$(B2)$ for a.a. $t \in [t_0, T]$, $f(t, \cdot)$ is nondecreasing.

In the scalar case, Hassan and Rzymowski, [10], gave sufficient conditions for the existence of a greatest and a least solution that improve both $(C1) - (C2) - (C3)$ and $(B1) - (B2) - (C3)$: they were able to replace $(C2)$ by

(HR) for a.a. $t \in [t_0, T]$ and all $x \in \mathbb{R}$, we have

$$\limsup_{y \to x^-} f(t, y) \le f(t, x) \le \liminf_{y \to x^+} f(t, y),$$

and, moreover, showed that the maximal (or greatest) solution is the biggest subfunction (see theorem 3.1 in [10]). Using a revision of Hassan and Rzymowski's arguments, López Pouso showed in [12] that (HR) may fail along a finite number of curves in the (t, x) plane, and therefore infinitely many discontinuity jumps in the wrong direction can be allowed.

In the monograph [7], Carl and Heikkilä replaced $(C2)$ by

(CH) f is non-negative and for each $(s, z) \in [t_0, T) \times \mathbb{R}$ there exist $\delta > 0$ and $\varepsilon > 0$ such that for a.a. $t \in [s, s + \delta]$ and all $x \in (z, z + \varepsilon]$ we have $\limsup_{y \to x^-} f(t, y) \le f(t, x)$ and for a.a. $t \in [s, s + \delta]$ and all $x \in [z, z + \varepsilon]$ we have $f(t, x) \le \liminf_{y \to x^+} f(t, y)$.

Even though the previous sketch is brief and rather incomplete, the reader can have an idea of the intensive and fruitful proccess of relaxation of Carathéodory conditions that is being developed by several people these years. Notwithstanding this enormous recent work, it seems that we are still very far from an optimal sufficient condition, since (1.1) is solvable for a lot of "strange" f's, as it is emphasized in chapter 12, theorem 3, in [11], and therefore any new existence criterion will be most welcome.

In this paper we are concerned with existence of Carathéodory solutions for (1.1) with non-negative right-hand sides. Our motivation is the main result in [13], which guarantees that (1.1) has a maximal and a minimal solution with positive derivative almost everywhere provided that $f : [t_0, T] \times [x_0, R] \longrightarrow [0, \infty)$ satisfies a certain boundedness condition and

$(P1)$ the composition $f(\tau(\cdot), \cdot)$ is measurable on $[x_0, R]$ whenever the function $\tau : [x_0, R] \to [t_0, T]$ is absolutely continuous and nondecreasing;

$(P2)$ for a.e. $x \in [x_0, R]$, $f(\cdot, x)$ is nonincreasing on $[t_0, T]$.

If we compare $(P1) - (P2)$ with $(B1) - (B2)$ we see that the roles of t and x are interchanged somehow. Since $(B1) - (B2) - (C3)$ were improved (in the scalar case) to $(C1) - (HR) - (C3)$, it is reasonable to think that $(P1) - (P2)$ could be extended in an analogous way. This will be achieved by looking at the problem from a new point of view, completely different from that of [13], and obtaining, by the way, some useful and applicable conclusions even for continuous nonlinearities.

This paper is organized as follows: in section 2 we take into account that locally invertible solutions of (1.1) solve a "reciprocal" problem and we point out some interesting computational and theoretical consequences of it. In section 3 we derive new local existence conditions for (1.1) by adapting recent results to the reciprocal problem and then translating the corresponding conditions to the framework of (1.1); a remarkable achievement is that no upper bound on f will be needed. In section 4 we present an extra condition that ensures solvability of (1.1) on a given interval. In section 5 we extend the results derived in sections 3 and 4 to cover right-hand sides with a more complicated behavior with respect to t. Finally, we give some examples in section 6 concerning the applicability of our results. Although no continuity assumption on f will be required, a surprising corollary of our theorem 2.1 is that (3.1) has a local solution if $f : [t_0, T] \times [x_0, R] \to \mathbb{R}$ satisfies

- for all $t \in [t_0, T]$, $f(t, \cdot)$ is measurable;
- for a.a. $x \in [x_0, R]$, $f(\cdot, x)$ is continuous;

- there exists $M \in L^1(x_0, R)$ such that for all $t \in [t_0, T]$ and a.a. $x \in [x_0, R]$
$$0 < f(t,x) \le M(x),$$

which seem to be a type of "inverse" Carathéodory conditions.

2 The inverse of a solution solves a reciprocal ODE

Consider the problem

$$x'(t) = f(t, x(t)), \quad t \ge t_0, \quad x(t_0) = x_0, \tag{2.2}$$

where $f : [t_0, T] \times [x_0, R] \to \mathbb{R}$, $T > t_0$ and $R > x_0$.

Definition 2.1 *A solution of (2.2) is an absolutely continuous function $x : [t_0, T_x] \to \mathbb{R}$, where $t_0 < T_x \le T$, such that $x([t_0, T_x]) \subset [x_0, R]$, $x'(t) = f(t, x(t))$ for a.a. $t \in [t_0, T_x]$ and $x(t_0) = x_0$.*

As the reader can infer from this definition, we shall concentrate on solutions which are defined on the right of t_0. There are two main reasons for not considering solvability on the left as well: first, in the discontinuous case sufficient conditions for existence on the right of t_0 are usually different from the ones which ensure existence on the left (although they are the same in the continuous case); second, a simple change of variables turns one of these problems into the other one and reveals the precise way to adapt any existence result from one to other case.

For $f : [t_0, T] \times [x_0, R] \to \mathbb{R}$ we define $\tilde{f} : [x_0, R] \times [t_0, T] \to \mathbb{R}$ as

$$\tilde{f}(r, y) = \begin{cases} \dfrac{1}{f(y, r)}, & \text{if} \quad f(y, r) \ne 0, \\[2ex] 0, & \text{if} \quad f(y, r) = 0, \end{cases}$$

and we consider the reciprocal problem

$$y'(r) = \tilde{f}(r, y(r)), \quad r \ge x_0, \quad y(x_0) = t_0. \tag{2.3}$$

Note that $\tilde{\tilde{f}} = f$.

The relation between (2.2) and (2.3) is established by the following fundamental result.

Theorem 2.1 *If $y : [x_0, R_y] \to \mathbb{R}$, where $x_0 < R_y \le R$, is a solution of (2.3) and, moreover, $y'(r) > 0$ for a.a. $r \in [x_0, R_y]$, then $y^{-1} : [t_0, y(R_y)] \to \mathbb{R}$ is a solution of (2.2).*

This theorem's proof is a consequence of the following proposition 2.2 and corollary 2.4. A proof for the next proposition can be found in [5].

Proposition 2.2 *Let $I = [a, b]$ and $J = [c, d]$ be a pair of nontrivial intervals and let $x : I \to J$ be one-to-one and onto and absolutely continuous on I.*

Then $x^{-1} : J \to I$ is absolutely continuous on J if and only if $x'(t) \neq 0$ for a.a. $t \in I$.

Moreover, if $x^{-1} \in AC(J)$ then we have that

$$(x^{-1})'(r) = \frac{1}{x'(x^{-1}(r))} \quad \text{for a.a. } r \in J.$$

The following proposition follows rightaway from theorem 38.2 in [14].

Proposition 2.3 *Let $g : [a, b] \to [c, d]$ be an absolutely continuous function and let $N \subset [c, d]$ be a null measure set.*

If $g'(t) \neq 0$ for a.a. $t \in [a, b]$ then $g^{-1}(N)$ is a null measure set.

Corollary 2.4 *Let $g : [a, b] \to [c, d]$ be an absolutely continuous function such that $g'(t) \neq 0$ for a.a. $t \in [a, b]$ and $h_1,\ h_2 : [c, d] \setminus N \to \mathbb{R}$, where N is a null set.*

If $h_1(x) = h_2(x)$ for a.a. $x \in [c, d] \setminus N$ then $h_1(g(t)) = h_2(g(t))$ for a.a. $t \in [a, b]$.

Even though it is simple, we shall give the proof of theorem 2.1 for completeness and for the convenience of the reader:

Proof of theorem 2.1. By proposition 2.2, for a.a. $t \in [t_0, y(R_y)]$ we have

$$(y^{-1})'(t) = \frac{1}{y'(y^{-1}(t))} > 0. \tag{2.4}$$

On the other hand, for a.a. $r \in [x_0, R_y]$ we have

$$0 < y'(r) = \tilde{f}(r, y(r)) = \frac{1}{f(y(r), r)},$$

and then, applying the result of corollary 2.4 to $h_1 = y'$, $h_2 = 1/f(y(\cdot), \cdot)$, and $g = y^{-1}$, we obtain for a.a. $t \in [t_0, y(R_y)]$

$$0 < y'(y^{-1}(t)) = \frac{1}{f(y(y^{-1}(t)), y^{-1}(t))} = \frac{1}{f(t, y^{-1}(t))},$$

which, together with (2.4), shows that y^{-1} solves (2.2) on $[t_0, y(R_y)]$. □

Now we include another consequence of proposition 2.3 that will be useful in next sections.

Corollary 2.5 *Let $N_1 \subset [a, b]$ and $N_2 \subset [c, d]$ be two null measure sets and let $f_1,\ f_2 : [a, b] \times [c, d] \to \mathbb{R}$ be such that for all $t \in [a, b] \setminus N_1$ and for all $x \in [c, d] \setminus N_2$*

$$f_1(t, x) = f_2(t, x).$$

If $x : [a_1, b_1] \subset [a, b] \to [c, d]$ is an absolutely continuous function such that $x'(t) \neq 0$ for a.a. $t \in [a_1, b_1]$, then x is a solution of problem

$$x'(t) = f_1(t, x(t)) \quad \textit{for a.a. } t \in [a_1, b_1], \quad x(t_0) = x_0,$$

if and only if x is a solution of problem

$$x'(t) = f_2(t, x(t)) \quad \textit{for a.a. } t \in [a_1, b_1], \quad x(t_0) = x_0.$$

First applications of theorem 2.1 (and the ideas behind its proof)

1. Exact computation of the inverse. The explicit expression of the inverse can be found if the differential equation in (2.3) is good enough. As an example, consider the problem (2.2) with $t_0 = 0 = x_0$ and

$$f(t,x) = \begin{cases} \dfrac{1}{t + x^2}, & \text{if } \quad (t,x) \neq (0,0), \\ \\ 0, & \text{if } \quad (t,x) = (0,0). \end{cases}$$

We note that the differential equation does not correspond with any of the usual types of elementary integrable equations, such as Bernoulli, Riccati, and so on.

However, in this case, problem (2.3) is linear and

$$y(r) = 2\mathrm{e}^r - r^2 - 2r - 2 \quad \text{for all } r \in [0, +\infty),$$

defines its unique solution. Moreover $y'(r) > 0$ for all $r \in (0, \infty)$ which implies, by theorem 2.1, that $y^{-1} : [0,\infty) \to [0,\infty)$ solves (2.2).

In general, the explicit expression of y^{-1} will be impossible to obtain (this is also a limitation one encounters when solving separable equations), however if $\{x_j\}_{j=1}^n$ is a subset of the domain of y, then the points

$$(y(x_1), x_1), (y(x_2), x_2), \ldots, (y(x_n), x_n),$$

lie exactly on the graph of a solution of (2.2), which gives very precise information from the point of view of numerical analysis. This fact is especially important in our example because it is singular, in the sense that $|f|$ goes to infinity as (t,x) approaches the initial condition $(0,0)$, and hence direct numerical resolution of (2.2) becomes harder than usual.

2. An existence result for (continuous) singular problems. As a corollary of Peano's theorem we have the following existence result, whose proof requires essentially the same arguments as that of theorem 2.1:

Theorem 2.6 *Let B be a ball centered at $(x_0, t_0) \in \mathbb{R}^2$.*

If $\tilde{f}$ is continuous on B and is positive (or negative) on $B \setminus \{(x_0, t_0)\}$,

then (2.2) has at least one absolutely continuous solution.

Note that f needs not be bounded on any neighbourhood of (t_0, x_0) in last theorem.

3. An alternative version of Lipschitz uniqueness criterion. If f is continuous in a neighbourhood of (t_0, x_0) then Peano's theorem ensures the existence of a local solution through (t_0, x_0) for $x' = f(t, x)$. Tipically, one would try to check wether a local Lipschitz condition in x is satisfied in order to ensure that the solution is locally unique. However this is not the case for, for instance, the problem

$$x' = f(t, x) = \sqrt{|x|} + \cos t, \quad x(0) = 0. \tag{2.5}$$

Anyway, we have that $f(0, 0) = 1 > 0$ and f is Lipschitz continuous with respect to t, and therefore $\tilde{f}(r, y) = 1/(\sqrt{|r|} + \cos y)$ is Lipschitz continuous with respect to y on a neighbourhood of $(0, 0)$. Hence the corresponding (2.3) problem has a unique solution. Furthermore, since f is positive in a neighbourhood of $(0, 0)$ then any solution of (2.5) defines locally a solution for the reciprocal problem (2.3), and therefore (2.5) has a unique solution.

This example falls inside the scope of

Theorem 2.7 *If $f = f(t, x)$ is continuous on a neighbourhood of $(t_0, x_0) \in \mathbb{R}^2$ and $f(t_0, x_0) \neq 0$, then the problem*

$$x' = f(t, x), \quad x(t_0) = x_0,$$

has a unique local solution provided that

either f is Lipschitz continuous in x on a neighbourhood of (t_0, x_0),

or f is Lipschitz continuous in t on a neighbourhood of (t_0, x_0).

Concerning theorem 2.7 it is well known that in case $f(t_0, x_0) = 0$ then uniqueness cannot be deduced from Lipschitz continuity with respect to t. Take for instance the problem

$$x' = \sqrt{|x|}, \quad t \geq 0, \quad x(0) = 0.$$

On the other hand, we point out that $f(t_0, x_0) \neq 0$ alone is not sufficient to ensure local uniqueness: the functions $x_1(t) = t$ and $x_2(t) = t^2/4 + t$, $t \geq 0$, are both solutions of

$$x' = \sqrt{|x - t|} + 1, \quad t \geq 0, \quad x(0) = 0.$$

Note however that the right-hand side is not Lipschitz continuous, neither in x nor in t, on any neighbourhood of $(0, 0)$.

3 Local existence. Singular problems

In this section we give sufficient conditions for the existence of a local solution of problem (2.2) in absence of upper bounds for f.

First we introduce some notation. We denote by $AC([a,b])$ the set of all real functions which are absolutely continuous on the interval $[a,b]$.

Definition 3.1 *For a subset $Y \subset AC([t_0,T_0])$, $t_0 < T_0 \le T$, we say that $x_* \in Y$ is the minimal minimal solution of (2.2) in Y if x_* is a solution of (2.2) and $x_*(t) \le x(t)$ for all $t \in [t_0,T_0]$ and for any other solution $x \in Y$. We define the maximal solution of (2.2) in Y by reversing the inequalities. When both, the minimal and the maximal solutions of (2.2) in Y exist, we call them extremal solutions in Y.*

We shall also need the following definition:

Definition 3.2 *We say that x_- is a subfunction on $[t_0,T_0] \subset [t_0,T]$ for the problem (2.2) if $x_- \in AC([t_0,T_0])$, $x_-(t_0) = x_0$, and*

$$(x_-)'(t) \le f(t, x_-(t)) \quad \text{for a.a. } t \in [t_0,T_0].$$

If $T_0 = T$ we say that x_- is a subfunction.

Analogously, we say that x_+ is an upperfunction on $[t_0,T_0] \subset [t_0,T]$ for the problem (2.2) if $x_+ \in AC([t_0,T_0])$, $x_+(t_0) = x_0$, and

$$(x_+)'(t) \ge f(t, x_+(t)) \quad \text{for a.a. } t \in [t_0,T_0].$$

In case $T_0 = T$ we say that x_+ is an upperfunction.

We shall assume that for the given right-hand side $f : [t_0,T] \times [x_0,R] \to \mathbb{R}$ there exists a null measure set $N \subset [x_0,R]$ such that the following conditions hold:

$(f1)$ for all $t \in [t_0,T]$, $f(t,\cdot)$ is measurable on $[x_0,R]$;

$(f2)$ for all $x \in [x_0, R] \setminus N$,

$$\liminf_{s \to t^-} f(s,x) \geq f(t,x) \quad \text{for all } t \in (t_0, T],$$

$$f(t,x) \geq \limsup_{s \to t^+} f(s,x) \quad \text{for all } t \in [t_0, T);$$

$(f3)$ there exists $M \in L^1(x_0, R)$ such that for all $x \in [x_0, R] \setminus N$ and all $t \in [t_0, T]$,

$$0 < \frac{1}{M(x)} \leq f(t,x).$$

Remark 3.1 *The right-hand side in the problem*

$$x' = \sqrt{x}, \; x(0) = 0, \tag{3.6}$$

satisfies $(f1)$-$(f2)$-$(f3)$ *on* $[0,1] \times [0,1/4]$ *with* $N = \{0\}$ *and* $M(x) = 1/\sqrt{x}$.

Note that (3.6) has infinitely many solutions defined on $[0,1]$ *but its unique solution with positive derivative almost everywhere is* $x(t) = t^2/4$, $t \in [0,1]$.

Since we shall restrict our attention to solutions which have positive derivative almost everywhere, we loose no generality if we consider $N = \emptyset$ *in* $(f1)$-$(f2)$-$(f3)$. *Indeed, if it were not the case it would suffice to define* $\bar{f} : [t_0, T] \times [x_0, R]$ *as*

$$\bar{f}(t,x) = \begin{cases} f(t,x), & \text{if } \; x \in [x_0, R] \setminus N, \\ \\ 1, & \text{if } \; x \in N, \end{cases}$$

and $\bar{M} : [x_0, R] \to \mathbb{R}$ *as*

$$\bar{M}(x) = \begin{cases} M(x), & \text{if } \; x \in [x_0, R] \setminus N, \\ \\ 1, & \text{if } \; x \in N. \end{cases}$$

It is easy to check that $\bar{f}$ and $\bar{M}$ satisfy (f1), (f2) and (f3) with $N = \emptyset$.

Moreover

$$f(t,x) = \bar{f}(t,x) \quad \textit{for all } t \in [t_0, T] \textit{ and all } x \in [x_0, R] \setminus N,$$

and therefore, by corollary 2.5, the problem (2.2) and the problem

$$x'(t) = \bar{f}(t, x(t)), \quad t \geq t_0, \quad x(t_0) = x_0,$$

have the same solutions with positive derivative almost everywhere (but not neccesarily the same solutions, as the reader can verify with (3.6)).

As a consequence of theorem 2.1 and recent results for the problem (2.3) we will prove the following theorem.

Theorem 3.1 *Suppose that for $f : [t_0, T] \times [x_0, R] \to \mathbb{R}$ there exists a null measure set $N \subset [x_0, R]$ such that $(f1)$-$(f2)$-$(f3)$ are satisfied.*

Then there exists $T_0 \in (t_0, T]$ such that (2.2) has extremal solutions in the set $Y_0 = \{x \in AC([t_0, T_0]) : x'(t) > 0 \quad \textit{for a.a. } t \in [t_0, T_0]\}$. Moreover, if x_* and x^* stand respectively for the minimal and the maximal solution in Y_0, then for all $t \in [t_0, T_0]$ we have

$$x^*(t) = \max\left\{x_-(t) \,/\, x_- \in Y_0 \textit{ is a subfunction on } [t_0, T_0]\right\},$$

$$x_*(t) = \min\left\{x_+(t) \,/\, x_+ \in Y_0 \textit{ is an upperfunction on } [t_0, T_0]\right\}.$$

Proof. By remark 3.1 we can assume that $N = \emptyset$ in $(f1)$-$(f2)$-$(f3)$.
Claim 1. Problem (2.2) has at least one solution in Y_0. First we will prove that the function $\tilde{f}$ satisfies the following properties:
i) For all $y \in [t_0, T]$, $\tilde{f}(\cdot, y)$ is measurable on $[x_0, R]$.

Let $y \in [t_0, T]$ be fixed. By condition $(f3)$ with $N = \emptyset$, we have that

$$f(y, x) > 0 \quad \text{for all } x \in [x_0, R], \tag{3.7}$$

and therefore $\tilde{f}(x, y) = 1/f(y, x)$ for all $x \in [x_0, R]$ and $\tilde{f}(\cdot, y)$ is measurable by virtue of $(f1)$.

ii) For all $r \in [x_0, R]$,

$$\limsup_{z \to y^-} \tilde{f}(r, z) \le \tilde{f}(r, y) \quad \textit{for all } y \in (t_0, T],$$

$$\tilde{f}(r, y) \le \liminf_{z \to y^+} \tilde{f}(r, z) \quad \textit{for all } y \in [t_0, T).$$

From $(f2)$ and (3.7) it follows that for all $(r, y) \in [x_0, R] \times (t_0, T]$

$$\limsup_{z \to y^-} \tilde{f}(r, z) = \limsup_{z \to y^-} \frac{1}{f(z, r)} = \frac{1}{\liminf_{z \to y^-} f(z, r)} \le \frac{1}{f(y, r)} = \tilde{f}(r, y),$$

and in an analogous way we can prove the other inequality.
iii) For all $r \in [x_0, R]$ and all $y \in [t_0, T]$ we have $|\tilde{f}(r, y)| \le M(r)$. Immediate from $(f3)$ with $N = \emptyset$.

Now we extend $\tilde{f}$ to $[x_0, R] \times \mathbb{R}$ by defining $\hat{f} : [x_0, R] \times \mathbb{R} \to \mathbb{R}$ as

$$\hat{f}(r, y) = \begin{cases} 0, & \text{if} \quad y < t_0, \\ \tilde{f}(r, y), & \text{if} \quad t_0 \le y \le T, \\ M(r), & \text{if} \quad y > T. \end{cases}$$

By properties $i)$, $ii)$ and $iii)$ we have that $\hat{f}$ satisfies the conditions of theorem 3.1 in [10], and hence the (non-local) problem

$$y'(r) = \hat{f}(r, y(r)) \quad \text{for a.a. } r \in [x_0, R], \quad y(x_0) = t_0, \tag{3.8}$$

has extremal solutions. Furthermore, the maximal solution is the biggest subfunction and the minimal solution is the smallest upperfunction.

If $y : [x_0, R] \to \mathbb{R}$ is a solution of (3.8) we have that $y(r) \ge t_0$ for all $r \in [x_0, R]$ because $\hat{f} \ge 0$. Therefore, in view of (3.7), for any solution y of (3.8) and for a.a. $r \in [x_0, R]$ we have

$$y'(r) = \left\{ \begin{array}{ll} \tilde{f}(r, y(r)), & \text{if } t_0 \le y(r) \le T \\ \\ M(r), & \text{if } y(r) > T \end{array} \right\} > 0. \tag{3.9}$$

Now put

$$T_0 = \min\{T, y_*(R)\},$$

where y_* is the minimal solution of (3.8). We have by (3.9) that if $y : [x_0, R] \to \mathbb{R}$ is a solution of (3.8) then

$$y'(r) = \tilde{f}(r, y(r)) \quad \text{for a.a. } r \in [x_0, y^{-1}(T_0)],$$

hence, by theorem 2.1, we have that $y^{-1} : [t_0, T_0] \to \mathbb{R}$ is a solution of problem (2.2) which, moreover, belongs to Y_0, as follows from proposition 2.2.

Claim 2. Problem (2.2) has extremal solutions in Y_0. We know that (3.8) has a minimal solution y_* and a maximal one y^* whose respective inverses are solutions of (2.2) in Y_0. Moreover, if y is a solution of (3.8) then y^{-1} defines a solution of (2.2) in Y_0 and $(y_*)^{-1} \geq y^{-1} \geq (y^*)^{-1}$ on $[t_0, T_0]$. Therefore, in order to ensure that $x^* = (y_*)^{-1}$ is the maximal solution of (2.2) in Y_0 and $x_* = (y^*)^{-1}$ is the minimal one, it suffices to prove that any solution of (2.2) in Y_0 is the inverse of a solution of (3.8). Indeed, let $x \in Y_0$ be a solution of (2.2). If $x(T_0) < R$ then x^{-1} is a solution of (3.8) on the interval $[x_0, x(T_0)] \subsetneq [x_0, R]$. Since the problem

$$y' = \hat{f}(r, y), \quad r \in [x(T_0), R], \quad y(x(T_0)) = T_0,$$

satisfies the conditions of theorem 3.1 in [10], then it has at least one solution $Y : [x(T_0), R] \to \mathbb{R}$. Hence the function $y : [x_0, R] \to \mathbb{R}$ defined as

$$y(r) = \begin{cases} x^{-1}(r), & \text{if } r \in [x_0, x(T_0)], \\ \\ Y(r), & \text{if } r \in (x(T_0), R], \end{cases}$$

is a solution of (3.8).

In case $x(T_0) = R$ then x^{-1} is defined on $[x_0, R]$ and solves (3.8).

Claim 3. If x_* denotes the minimal solution in Y_0 of (2.2) then $x_*(t) = \min \{x_+(t) \,/\, x_+ \in Y_0 \textit{ is an upperfunction on } [t_0, T_0]\}$. Since x_* is an upperfunction and $x_* \in Y_0$ it suffices to see that if $x \in Y_0$ is an upperfunction on $[t_0, T_0]$ for (2.2) then $x_* \leq x$. To accomplish this we define $y(r) = x^{-1}(r)$ for $r \in [x_0, x(T_0)]$ and, following similar arguments to those in the proof of theorem 2.1, on can prove that y is a subfunction on $[x_0, x(T_0)]$ for (3.8), therefore if y^* denotes the maximal solution of (3.8) we have

$$y \leq y^* \quad \text{on } [x_0, x(T_0)] \Leftrightarrow x = y^{-1} \geq (y^*)^{-1} = x_* \quad \text{on } [t_0, T_0].$$

An analogous argument leads to the expression

$$x^*(t) = \max \{x_-(t) \,/\, x_- \in Y_0 \text{ is a subfunction on } [t_0, T_0]\}.$$

$\square$

4 Global existence. Bounded right-hand sides

The conditions $(f1)$-$(f2)$-$(f3)$ are not sufficient to ensure the existence of a global solution on the whole interval $[t_0, T]$ for problem (2.2), as we show in the following simple example:

Example 4.1 *Take $f(t,x) = 1$ for all $(t,x) \in [0,2] \times [0,1]$. The unique solution of*

$$x'(t) = f(t, x(t)), \quad t \geq 0, \quad x(0) = 0,$$

is given by $x(t) = t$, $t \in [0,1] \subsetneq [0,2]$.

In this section we give sufficient conditions for the existence of extremal solutions for the problem (2.2) in the set

$$Y = \{x \in AC([t_0, T]) \ : \ x'(t) > 0 \text{ for a.a. } t \in [t_0, T]\}. \tag{4.10}$$

If $x \in Y$ is a solution of (2.2) then in particular x is a global solution, since x is defined on the whole interval $[t_0, T]$.

Theorem 4.1 *Assume that there exists a null measure set $N \subset [x_0, R]$ such that the function $f : [t_0, T] \times [x_0, R] \to \mathbb{R}$ satisfies $(f1)$-$(f2)$-$(f3)$. Assume also that*

(f4) there exists $m \in L^1(x_0, R)$ such that

(f4-1) for all $x \in [x_0, R] \setminus N$ and all $t \in [t_0, T]$,

$$f(t,x) \leq \frac{1}{m(x)};$$

(f4-2) $\displaystyle\int_{x_0}^{R} m(s)ds \geq T - t_0.$

Then problem (2.2) has extremal solutions in Y, where the set Y is defined in (4.10). Moreover, if x_* and x^* stand respectively for the minimal *and the maximal solution in Y, then for all $t \in [t_0, T_0]$ we have*

$$x^*(t) = \max\{x_-(t) \,/\, x_- \in Y \text{ is a subfunction}\},$$

$$x_*(t) = \min\{x_+(t) \,/\, x_+ \in Y \text{ is an upperfunction}\}.$$

Proof. By remark 3.1 we can suppose that $N = \emptyset$. Now it suffices to repeat the proof of theorem 3.1 and to show that

$$T_0 := \min\{T, y_*(R)\} = T, \tag{4.11}$$

where y_* is the minimal solution of (3.8). To see this note that if y is a solution of (3.8) then for a.a. $r \in [x_0, R]$ we have

$$y'(r) = \left\{ \begin{array}{ll} \tilde{f}(r, y(r)), & \text{if } t_0 \le y(r) \le T \\ \\ M(r), & \text{if } y(r) > T \end{array} \right\} \ge m(r).$$

Therefore, using condition $(f4)$, we compute

$$y(R) = t_0 + \int_{x_0}^{R} y'(s)ds \ge t_0 + \int_{x_0}^{R} m(s)ds \ge T,$$

and (4.11) is proven. □

The function $f : [0, 1] \times [0, 2] \to \mathbb{R}$ given by

$$f(t, x) = \begin{cases} 1, & \text{if } \ 0 \le t \le \frac{1}{2}, \\ \\ 2, & \text{if } \ \frac{1}{2} < t \le 1, \end{cases}$$

does not satisfy $(f2)$. Nevertheless f fulfills the conditions of the following more general theorem, whose proof is based on some ideas contained in the proof of theorem 2.1.4. in [7].

Theorem 4.2 *We suppose that $f : [t_0, T] \times [x_0, R] \to \mathbb{R}$ satisfies (f4) and for all $(t, x) \in [t_0, T) \times [x_0, R)$ there exist $\delta = \delta(t, x) > 0$ and $\varepsilon = \varepsilon(t, x) > 0$ such that the restriction $f : [t, t + \delta] \times [x, x + \varepsilon] \to \mathbb{R}$ satisfies (f1), (f2) and (f3).*

Then problem (2.2) has extremal solutions in Y. Moreover, if x_* and x^* stand respectively for the minimal and the maximal solution in Y, then for all $t \in [t_0, T_0]$ we have

$$x^*(t) = \max\{x_-(t) \,/\, x_- \in Y \ \textit{is a subfunction}\},$$

$$x_*(t) = \min\{x_+(t) \,/\, x_+ \in Y \ \textit{is an upperfunction}\}.$$

Proof. Let δ and ε be such that $f : [t_0, t_0+\delta] \times [x_0, x_0+\varepsilon] \to \mathbb{R}$ satisfies $(f1)$-$(f2)$-$(f3)$, and moreover $\displaystyle\int_{x_0}^{x_0+\varepsilon} m(s)ds \geq \delta$. Then, theorem 4.1 ensures that problem (2.2) has extremal solutions on $[t_0, t_0+\delta]$ with positive derivative a.a. on $[t_0, t_0+\delta]$.

We define t_1 as the supreme of $t \in (t_0, T]$ such that the problem (2.2) has extremal solutions in the set of absolutely continuous functions on the interval $[t_0, t]$ with positive derivative for a.a. $t \in [t_0, t]$. We have that $t_1 = T$. Indeed, if $t_1 < T$ we have two possibilities:
Case 1) The maximal solution x^* on $[t_0, t_1]$ satisfies that $x^*(t_1) < R$. In this case we can repeat the above reasoning and obtain a continuation of x^*, which contradicts with the choice of t_1.
Case 2) The maximal solution x^* on $[t_0, t_1]$ satisfies that $x^*(t_1) = R$. In this case we have that $y = (x^*)^{-1}$ satisfies

$$y'(r) = \tilde{f}(t, y(r)) \geq m(r) \quad \text{for a.a. } r \in [x_0, R],$$

and by $(f4)$ we have

$$t_1 = y(R) = y(x_0) + \int_{x_0}^{R} y'(r)dr \geq T,$$

a contradiction. □

5 A generalization of the previous results

In this section we prove the existence of local and global solutions for the problem

$$x'(t) = l(t)f(t, x(t)), \quad t \geq t_0, \quad x(t_0) = x_0, \tag{5.12}$$

where $l : [t_0, T] \to \mathbb{R}$ and $f : [t_0, T] \times [x_0, R] \to \mathbb{R}$, with $T > t_0$ and $R > x_0$.

The reciprocal problem, in the sense of theorem 2.1, is the following one

$$y'(r) = q(y(r))\tilde{f}(r, y(r)), \quad r \geq x_0, \quad y(x_0) = t_0, \tag{5.13}$$

where $q : [t_0, T] \to \mathbb{R}$ is given by

$$q(y) = \begin{cases} \dfrac{1}{l(y)}, & \text{if} \quad l(y) \neq 0, \\ \\ 0, & \text{if} \quad l(y) = 0, \end{cases}$$

and $\tilde{f}$ defined in section 2.

If l satisfies

($l0$) $l \in L^1(t_0, T)$ and $l(t) > 0$ for a.a. $t \in [t_0, T]$,

then q satisfies

($q0$) $\dfrac{1}{q} \in L^1(t_0, T)$ and $q(y) > 0$ for a.a. $y \in [t_0, T]$,

and it is possible to define

$$Q(y) := t_0 + \int_{t_0}^{y} \frac{1}{q(s)} ds \quad \text{for all } y \in [t_0, T]. \tag{5.14}$$

Note that Q is increasing and absolutely continuous on $[t_0, T]$, and so is Q^{-1} on its domain.

Now we consider the problem

$$z'(r) = \tilde{f}(r, Q^{-1} \circ z(r)), \quad r \geq x_0, \quad z(x_0) = t_0. \tag{5.15}$$

In the following proposition we establish an equivalence between problems (5.12), (5.13) and (5.15). Its proof is based on theorem 2.1 and the chain rule for absolutely continuous functions (the reader is refered to lemma 2 and remark 3 in [5] for a proof based on theorems 9.3 and 38.4 in [14]).

Proposition 5.1 *Assume that l satisfies* ($l0$) *and let f be a function defined on $[t_0, T] \times [x_0, R]$.*

I) If $x : [t_0, T_0] \to \mathbb{R}$, $t < T_0 \leq T$, is a solution of (5.12) with $x'(t) > 0$ for a.a. $t \in [t_0, T_0]$ then $y := x^{-1} : [x_0, x(T_0)] \to \mathbb{R}$ is a solution of (5.13) *with $y'(r) > 0$ for a.a. $r \in [x_0, R_y]$, and then $z := Q \circ y : [x_0, R_y] \to \mathbb{R}$ is a solution of (5.15) with $z'(r) > 0$ for a.a. $r \in [x_0, R_y]$;*

II) conversely, if $z : [x_0, R_z] \to \mathbb{R}$, with $x_0 < R_z \leq R$, is a solution of (5.15) and moreover $z'(r) > 0$ for a.a. $r \in [x_0, R_z]$, then $y := Q^{-1} \circ z : [x_0, R_z] \to \mathbb{R}$ is a solution of (5.13) with $y'(r) > 0$ for a.a. $r \in [x_0, R_z]$, and then $x := y^{-1} : [t_0, y(R_z)] \to \mathbb{R}$ is a solution of (5.12) with $x'(t) > 0$ for a.a. $t \in [t_0, y(R_z)]$.

In the following theorem we improve theorem 2.1 to cover problem (5.12).

Theorem 5.2 *Suppose that for $f : [t_0, T] \times [x_0, R] \to \mathbb{R}$ there exists a null measure set $N \subset [x_0, R]$ such that $(f1)$-$(f2)$-$(f3)$ are satisfied. Suppose also that $l \in L^1(t_0, T)$ satisfies $(l0)$.*

Then there exists $T_0 \in (t_0, T]$ such that (5.12) has extremal solutions in the set $Y_0 = \{x \in AC([t_0, T_0]) : x'(t) > 0 \quad \text{for a.a. } t \in [t_0, T_0]\}$. Moreover, if x_* and x^* stand respectively for the minimal and the maximal solution in Y_0, then for all $t \in [t_0, T_0]$ we have

$$x^*(t) = \max \{x_-(t) \,/\, x_- \in Y_0 \text{ is a subfunction on } [t_0, T_0] \text{ for (5.12)}\},$$

$$x_*(t) = \min \{x_+(t) \,/\, x_+ \in Y_0 \text{ is an upperfunction on } [t_0, T_0] \text{ for (5.12)}\}.$$

Sketch of the proof. The proof needs essentially the same arguments as that of theorem 2.1, but considering the modified problem (3.8) with

$$\hat{f}(r, z) = \begin{cases} 0, & \text{if } \; z < t_0, \\ \tilde{f}(r, Q^{-1}(z)), & \text{if } \; t_0 \leq z \leq Q(T), \\ M(r), & \text{if } \; z > Q(T). \end{cases}$$

Using theorem 3.1 in [10] one can prove that (3.8) has a minimal solution z_* and a maximal one z^*. Subsequently we define

$$T_0 := \min \left\{T, Q^{-1}(z_*(R))\right\}, \tag{5.16}$$

and, by means of proposition 5.1, one can deduce that $x_* = (z^*)^{-1} \circ Q$ is the minimal solution of (5.12) in Y_0 and that $x^* = (z_*)^{-1} \circ Q$ is the maximal one. $\square$

Exactly as we did for problem (2.2) we can give sufficient conditions for the existence of extremal solutions of (5.12) in the set Y, defined in (4.10).

Theorem 5.3 *In the conditions of theorem 5.2, suppose moreover that*

(f5) there exists $m \in L^1(x_0, R)$ such that

(f5-1) for all $x \in [x_0, R] \setminus N$ and for all $t \in [t_0, T]$,

$$f(t, x) \leq \frac{1}{m(x)};$$

(f5-2) $\displaystyle\int_{x_0}^{R} m(s)ds \geq \int_{t_0}^{T} l(s)ds.$

Then the problem (5.12) has extremal solutions in Y. Moreover, if x_* and x^* stand respectively for the minimal and the maximal solution in Y, then for all $t \in [t_0, T_0]$ we have

$$x^*(t) = \max\{x_-(t)\,/\,x_- \in Y \text{ is a subfunction for (5.12)}\}, \tag{5.17}$$

$$x_*(t) = \min\{x_+(t)\,/\,x_+ \in Y \text{ is an upperfunction for (5.12)}\}. \tag{5.18}$$

Sketch of the proof. Following the proof of theorem 5.2 step by step, it suffices to take into account the following property: if $z : [x_0, R] \to \mathbb{R}$ is a solution of (3.8) with $\hat{f}$ defined as in the proof of theorem 5.2, we have for a.a. $r \in [x_0, R]$ that

$$0 < m(r) \leq z'(r) = \hat{f}(r, z(r)) \leq M(r). \tag{5.19}$$

Therefore from (5.19) and from $(f5-2)$ we deduce that

$$z(R) = z(x_0) + \int_{x_0}^{R} z'(s)ds \geq t_0 + \int_{x_0}^{R} m(s)ds \geq t_0 + \int_{t_0}^{T} l(s)ds \geq Q(T),$$

and hence $T_0 = T$ in (5.16). $\square$

Finally we have the following more general result.

Theorem 5.4 *We suppose that $l : [t_0, T] \to \mathbb{R}$ satisfies $(l0)$, $f : [t_0, T] \times [x_0, R] \to \mathbb{R}$ satisfies $(f5)$ and for all $(t, x) \in [t_0, T) \times [x_0, R)$ there exist $\delta = \delta(t, x) > 0$ and $\varepsilon = \varepsilon(t, x) > 0$, such that the restriction $f : [t, t+\delta] \times [x, x+\varepsilon] \to \mathbb{R}$ satisfies $(f1)$-$(f2)$-$(f3)$.*

Then problem (5.12) has extremal solutions in Y. Moreover, if we denote by $x_* \in Y$ the minimal solution and by $x^* \in Y$ the maximal one, then x_* and x^* satisfy (5.17) and (5.18), respectively.

6 Examples and remarks

First we present one example of application of theorem 4.1.

Example 6.1 *Let $C \subset [0, 2]$ be a Cantor set with positive Lebesgue measure and denote by $\chi_C : [0, 2] \to \mathbb{R}$ its characteristic function.*

On the other hand, we define $\Phi : [0, 1] \to \mathbb{R}$ as

$$\Phi(t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(1 + sgn\left(t - \frac{m}{n}\right)\right) 2^{-m-n} \quad \text{for all } t \in [0, 1],$$

where $sgn(t) = \dfrac{t}{|t|}$ if $t \neq 0$ and $sgn(0) = 0$.

The reader can verify that Φ is increasing, $0 \leq \Phi(t) \leq 2$ for all $t \in [0, 1]$ and Φ is discontinuous at all rational points of $[0, 1]$.

Now, if we define $f : [0, 1] \times [0, 2] \to \mathbb{R}$ as

$$f(t, x) = \frac{1}{1 + \Phi(t)} + \chi_C(x) \quad \text{for all } (t, x) \in [0, 1] \times [0, 2],$$

we can easily see that f satisfies the assumptions of theorem 4.1 and then the problem

$$x'(t) = f(t, x(t)), \ t \geq 0, \quad x(0) = 0,$$

has extremal solutions in Y.

A next remark is concerned with the possibility of using our results in more general settings (which points out the possibility of improving our exitence results).

Remark 6.1 *The function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined as*

$$f(t,x) = \begin{cases} \dfrac{1}{n}, & \textit{if} \quad t = \dfrac{1}{n}, \ n = 1,2,3\ldots \\ \\ 1, & \textit{otherwise}, \end{cases}$$

is not in the conditions of theorem 3.1. Nevertheless if we define the null measure set $N_1 = \left\{ \dfrac{1}{n} \ : \ n = 1,2,3\ldots \right\} \subset [0,1]$ we have that

$$f(t,x) = 1 \quad \textit{for all } t \in [0,1] \setminus N_1 \textit{ and for all } x \in [0,1],$$

and therefore, by virtue of corollary 2.4, the problem

$$x'(t) = f(t, x(t)), \quad t \geq 0, \quad x(0) = 0,$$

and the problem

$$x'(t) = 1, \quad t \geq 0, \quad x(0) = 0,$$

have the same solutions with positive derivative almost everywhere. In particular $x(t) = t$ for all $t \in [0,1]$ defines the unique solution for both problems.

In the following example we show that if hypothesis $(f2)$ fails then problem 2.2 may have no solution.

Example 6.2 *Consider the problem*

$$x'(t) = f(t, x(t)), \ t \geq 0, \quad x(0) = 0,$$

where $f : [0,1] \times [0,1] \to \mathbb{R}$ is given by

$$f(t,x) = \begin{cases} \frac{1}{2}, & \textit{if} \quad 0 \le t \le x \le 1, \\ \\ 2, & \textit{if} \quad 0 \le x < t \le 1. \end{cases}$$

It is easy to verify that $(f2)$ does not hold in any rectangle $[t_0, t_1] \times [x_0, x_1] \subset [t_0, T] \times [x_0, R]$ and that there exists no solution for this problem.

Examples of the previous type were pointed out by Wend in [16].

On the other hand, if f does not satisfy the conditions of theorem 4.2 then (2.2) may have solutions in Y but not extremal solutions in Y.

Example 6.3 *Consider the problem*

$$x'(t) = f(t, x(t)), \ t \ge 0, \quad x(0) = 0,$$

where $f : [0,1] \times [0,1] \to \mathbb{R}$ is given by

$$f(t,x) = \begin{cases} \frac{1}{2}, & \textit{if} \quad 0 \le t \le x \le 2, \\ \\ \frac{2n^2-1}{2(n+1)n}, & \textit{if} \quad 0 \le \frac{n-1}{n} t \le x < \frac{n}{n+1} t \le 2, \ n = 1, 2, 3... \end{cases}$$

It is easy to see that f satisfies $(f1)$-$(f3)$-$(f4)$ with $m \equiv 1$ and $M \equiv 4$, but f does not satisfy $(f2)$ locally (in the sense of theorem 4.2). On the other hand, the solutions of this problem in Y are

$$x_n(t) = \frac{2n^2-1}{2(n+1)n} t \quad \textit{for all } t \in [0,1],$$

which defines an increasing functional sequence whose limit is not a solution. Hence there is not a maximal solution in this case.

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