

Initial value problems for singular and nonsmooth second order differential inclusions

Daniel C. Biles[†], J. Ángel Cid[‡], and Rodrigo López Pouso[‡]

[†]Department of Mathematics, Western Kentucky University, Bowling Green,
Kentucky 42101.

[‡]Departamento de Matemáticas, E.U.P. de Linares, Universidad de Jaén,
Alfonso X el Sabio, 28, 23700, Linares, Jaén, Spain.

[‡]Departamento de Análise Matemática, Faculdade de Matemáticas,
Universidade de Santiago de Compostela, 15782, Santiago de Compostela,
Spain.

1 Introduction

In this paper we prove an existence result concerning monotone $W^{2,1}$ solutions for the initial value problem

$$\begin{cases} x''(t) \in F(t, x(t)) & \text{for a.a. } t \in I := [0, T], \\ x(0) = x_0, \quad x'(0) = x_1, \end{cases} \quad (1.1)$$

where $T > 0$ is *a priori* fixed, $x_0, x_1 \in \mathbb{R}$, and $F : I \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ is a multi-valued mapping which may assume arbitrarily large sets of values, even around the initial condition $(0, x_0)$ (singularity), and needs not satisfy any usual Lipschitz or closed-graph condition with respect to the unknown (nonsmoothness) that one can find in recent references such as [3, 5]. Moreover, as an intermediate step towards our main result, we derive necessary and sufficient conditions for the existence of nonconstant solutions for (1.1) in the autonomous case.

Our existence result seems to be new even for differential equations, which correspond to differential inclusions with singleton-valued right-hand sides.

We follow the spirit of [1] in the construction of adequate selections of F , mixed with an argument already employed in [4, 7] which lets one extend existence results from autonomous to nonautonomous problems. A recent existence result proved in [8] for second order autonomous differential equations is also needed in our work.

2 Preliminaries

Consider first the autonomous problem

$$\begin{cases} x''(t) = f(x(t)) & \text{for a.a. } t \in I, \\ x(0) = x_0, \quad x'(0) = x_1. \end{cases} \quad (2.2)$$

The solutions of (2.2) that we are going to consider here are elements of the Sobolev space $W^{2,1}(I)$.

One can find in [8] the following necessary and sufficient conditions for the existence of nonconstant solutions of (2.2):

THEOREM 2.1 *Problem (2.2) has a nonconstant solution $x : I \rightarrow \mathbb{R}$ if and only if there exists $R > 0$ such that the following claims hold for at least one of the intervals*

$$J = \begin{cases} [x_0, x_0 + R], & \text{if } \text{sgn}(x_1) = 1, \\ [x_0 - R, x_0], & \text{if } \text{sgn}(x_1) = -1, \end{cases}$$

where $\text{sgn}(z) = z/|z|$ for $z \neq 0$ and $\text{sgn}(0) = \pm 1$:

1. $f \in L^1(J)$.
2. $x_1^2 + 2 \int_{x_0}^x f(r) dr > 0$ for a.a. $x \in J$.
3. $\frac{\max\{1, |f|\}}{\sqrt{x_1^2 + 2 \int_{x_0}^x f(r) dr}} \in L^1(J)$.
4. $\int_J \frac{dx}{\sqrt{x_1^2 + 2 \int_{x_0}^x f(r) dr}} \geq T$.

Moreover, in this case there exists an increasing solution if $\text{sgn}(x_1) = 1$ and a decreasing solution if $\text{sgn}(x_1) = -1$, and they are defined implicitly for all $t \in [0, T]$ by

$$\int_{x_0}^{x(t)} \frac{dr}{\sqrt{x_1^2 + 2 \int_{x_0}^r f(s) ds}} = \text{sgn}(x_1)t.$$

3 The autonomous differential inclusion

In this section we deal with the autonomous problem

$$\begin{cases} x''(t) \in F(x(t)) & \text{for a.a. } t \in I, \\ x(0) = x_0, \quad x'(0) = x_1, \end{cases} \quad (3.3)$$

where $F : \text{Dom}(F) \subset \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$.

DEFINITION 3.1 *Let $F : \text{Dom}(F) \subset \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ be a multivalued mapping. A function $f : [x_0, x_0 + R] \rightarrow \mathbb{R}$, $R > 0$, is an **admissible selection on the right of x_0** for F if f is a selection of $F|_{[x_0, x_0 + R]}$ and satisfies the following properties:*

- (i) $f \in L^1(x_0, x_0 + R)$.
- (ii) $x_1^2 + 2 \int_{x_0}^x f(r) dr > 0$ for a.a. $x \in [x_0, x_0 + R]$.
- (iii) $\frac{\max\{1, |f|\}}{\sqrt{x_1^2 + 2 \int_{x_0}^{\cdot} f(r) dr}} \in L^1(x_0, x_0 + R)$.
- (iv) $\int_{x_0}^{x_0 + R} \frac{dx}{\sqrt{x_1^2 + 2 \int_{x_0}^x f(r) dr}} \geq T$.

We define admissible selections on the left of x_0 in a similar way with intervals of the type $[x_0 - R, x_0]$ for some $R > 0$.

The relevance of admissible selections in our work comes from the following theorem. Note that it gives not only sufficient conditions for existence, but also necessary ones.

THEOREM 3.1 *The following claims hold:*

(i) Problem (3.3) has an increasing solution if and only if F has an admissible selection on the right of x_0 .

(ii) Problem (3.3) has a decreasing solution if and only if F has an admissible selection on the left of x_0 .

Proof. We only prove part (i) because part (ii) is similar.

Suppose that $x : I = [0, T] \rightarrow \mathbb{R}$ is an increasing solution of (3.3). Define

$$J := [x_0, x(T)] = [x_0, x_0 + R] \quad \text{for } R = x(T) - x_0 > 0$$

and define $f : J \rightarrow \mathbb{R}$ as

$$f(y) = \begin{cases} x''(x^{-1}(y)), & \text{if } x''(x^{-1}(y)) \text{ exists and } x''(x^{-1}(y)) \in F(y), \\ \text{any element of } F(y), & \text{in other case.} \end{cases}$$

Then f is a selection of F and, since x is a solution of problem (2.2) with this function f , by theorem 2.1 we have that f is an admissible selection on the right of x_0 of (3.3).

Conversely, if f is an admissible selection of F on the right of x_0 then by theorem 2.1 the problem (2.2) has an increasing solution x . Since f is in particular a selection of F we have that x is also an increasing solution of problem (3.3). \square

For the sake of clarity and completeness we give next some sufficient conditions for a multivalued mapping to have admissible selections on the right of x_0 . It is natural to look first at the greatest and the least selections, if they exist. Thus, for a given multivalued mapping $F : \text{Dom}(F) \subset \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ we define

$$i_F(x) := \inf F(x) \quad \text{and} \quad s_F(x) := \sup F(x) \quad \text{for all } x \in \text{Dom}(F), \quad (3.4)$$

where inf and sup are computed in the extended real line and, thus, they can assume the values $-\infty$ and $+\infty$, respectively. Note that if $f : J \rightarrow \mathbb{R}$ is a selection of $F|_J$ we immediately have

$$i_F(x) \leq f(x) \leq s_F(x) \quad \text{for all } x \in J.$$

The following proposition is a straightforward consequence of this definition, and it concerns the case when i_F or s_F are selections of F :

PROPOSITION 3.1 *Let $F : \text{Dom}(F) \subset \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ be a multivalued mapping and let i_F and s_F be defined as in (3.4).*

If there is $R > 0$ such that $i_F(x) \in F(x)$ (respectively, $s_F(x) \in F(x)$) for all $x \in [x_0, x_0 + R]$ and i_F (respectively, s_F) satisfies conditions (i) – (iv) in the definition (3.1), then i_F (respectively, s_F) is the least (respectively, the largest) admissible selection of F on $[x_0, x_0 + R]$.

In general, one cannot expect i_F and s_F to be selections of F . Next proposition is useful in those situations. Its proof is easy and so we omit it.

PROPOSITION 3.2 *Let $F : \text{Dom}(F) \subset \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ be a multivalued mapping and let i_F and s_F be defined as in (3.4).*

Assume there is $R > 0$ such that

- (i) $i_F, s_F \in L^1(x_0, x_0 + R)$.
- (ii) $x_1^2 + 2 \int_{x_0}^x i_F(r) dr > 0$ for a.a. $x \in [x_0, x_0 + R]$.
- (iii) $\frac{\max\{1, |i_F|, |s_F|\}}{\sqrt{x_1^2 + 2 \int_{x_0}^x i_F(r) dr}} \in L^1(x_0, x_0 + R)$.
- (iv) $\int_{x_0}^{x_0+R} \frac{dx}{\sqrt{x_1^2 + 2 \int_{x_0}^x s_F(r) dr}} \geq T$.

Then any measurable selection of $F|_{[x_0, x_0+R]}$ is an admissible selection of F on the right of x_0 .

The main drawback in our last proposition is that it does not directly provide us with a criteria of existence of admissible selections. In fact, to apply proposition 3.2 satisfactorily we also need to know that our multivalued mapping has measurable selections. Loosely speaking, we can say that measurable closed-valued mappings have measurable selections. In order to clarify our last statement we recall some definitions: let (X, \mathcal{M}) be a measurable space and Y

a topological space, we say that a multivalued mapping $F : X \rightarrow \mathcal{P}(Y) \setminus \{\emptyset\}$ is measurable when for all open $U \subset Y$ we have

$$F^{-1}(U) := \{x \in X : F(x) \cap U\} \in \mathcal{M};$$

also, a topological space Y is a Polish space if it is homeomorphic to a complete separable metric space. Now we are in a position to present a precise statement (see [11]) of the result mentioned above:

THEOREM 3.2 (KURATOWSKI-RYLL-NARDZEWSKI) *Let (X, \mathcal{M}) be a measurable space and Y be a Polish space.*

If $F : X \rightarrow \mathcal{P}(Y) \setminus \{\emptyset\}$ is measurable and assumes only closed values then F admits a measurable selection.

As an immediate consequence we have the following proposition:

PROPOSITION 3.3 *Assume that the conditions of proposition 3.2 hold for some $R > 0$.*

If $F : [x_0, x_0 + R] \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ is measurable and assumes only closed values then F has admissible selections on $[x_0, x_0 + R]$.

Remark. Proposition 3.3 needs that F be closed-valued, which implies that i_F and s_F are selections of F . Taking into account the remaining conditions in proposition 3.3, we can ensure that i_F and s_F are then admissible selections of F on the right of x_0 , which is not much more interesting than the result in proposition 3.1.

To take a more satisfactory profit of proposition 3.3 the reader must notice that F does not really need to assume closed values. In general, one has to look for a suitable closed-valued $G : \text{Dom}(F) \subset \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ such that $G(x) \subset F(x)$ for all $x \in \text{Dom}(F)$, and try to use proposition 3.3 with G instead of F . Obviously, every admissible selection of G will be an admissible selection of F .

For more information on measurable selections we refer the reader to [10, 11].

4 The nonautonomous differential inclusion

Let $F : [0, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$, $\text{sgn}(x_1) = 1$ and

$$\hat{X} = \{u \in \mathcal{C}([0, T]) : u(0) = x_0, u \text{ is nondecreasing}\}.$$

For each $u \in \hat{X}$ we define its ‘‘pseudoinverse’’ $\hat{u} : \mathbb{R} \rightarrow [0, T]$ as

$$\hat{u}(x) = \begin{cases} 0, & x < x_0, \\ \min u^{-1}(\{x\}), & x_0 \leq x \leq u(T), \\ T, & u(T) < x. \end{cases}$$

We notice that \hat{u} is nondecreasing, but not necessarily continuous. Moreover, if $u \in \hat{X}$ is increasing in I , then $\hat{u}(x) = u^{-1}(x)$ for all $x \in [x_0, u(T)]$.

Assume that for some $R > 0$ the following hypotheses hold:

(F1) For each $u \in \hat{X}$ the multifunction $F_u : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ defined as $F_u(\cdot) = F(\hat{u}(\cdot), \cdot)$ has an admissible selection on the right of x_0 $f_u : [x_0, x_0 + R] \rightarrow \mathbb{R}$.

(F2) There exists $M \in L^1(x_0, x_0 + R)$ such that for all $t \in I$ and all $x \in [x_0, x_0 + R]$ we have

$$\sup\{y : y \in F(t, x)\} \leq M(x).$$

(F3) For each $u, v \in \hat{X}$, the relation $u \leq v$ on I implies $f_u \leq f_v$ on $[x_0, x_0 + R]$.

The following is our main result.

THEOREM 4.1 *Suppose that conditions (F1), (F2) and (F3) hold for some $R > 0$. Then problem (1.1) with $\text{sgn}(x_1) = 1$ has an increasing solution.*

Proof. We define the operator $G : \hat{X} \rightarrow \hat{X}$ in the following way: for each $u \in \hat{X}$ the function Gu is given implicitly for all $t \in [0, T]$ by

$$\int_{x_0}^{Gu(t)} \frac{dx}{\sqrt{x_1^2 + 2 \int_{x_0}^x f_u(r) dr}} = t,$$

and in particular Gu is an increasing solution of

$$\begin{cases} x''(t) = f_u(x(t)) & \text{for a.a. } t \in I, \\ x(0) = x_0, \quad x'(0) = x_1. \end{cases} \quad (4.5)$$

Note also that $Gu(T) \leq x_0 + R$ because f_u is an admissible selection.

We claim that there exists $L > 0$ such that $|(Gu)'(t)| \leq L$ for all $t \in I$ and all $u \in \hat{X}$. To prove it, let $u \in \hat{X}$ be given and denote $x = Gu$; by (F2) for a.e. $t \in [0, t]$ we have that

$$x''(t) = f_u(x(t)) \leq M(x(t)),$$

and since $x' \geq 0$ on $[0, T]$, we have

$$x''(t)x'(t) \leq M(x(t))x'(t) \quad \text{for a.a. } t \in [0, T].$$

Now we integrate between 0 and $t \in [0, T]$ in the previous inequality and we apply theorems 9.3 and 38.4 in [9] to conclude that

$$x'^2(t) \leq x_1^2 + 2 \int_{x_0}^{x(t)} M(s)ds,$$

and since $x_0 \leq x(t) \leq x_0 + R$ for all $t \in [0, T]$ and $x' \geq 0$ on $[0, T]$, we finally obtain

$$0 \leq x'(t) \leq \sqrt{x_1^2 + 2 \int_{x_0}^{x_0+R} M(s)ds} =: L \quad \text{for all } t \in [0, T].$$

Therefore, it is clear that if we define

$$X = \{u \in \hat{X} : |u(t) - u(s)| \leq L|t - s| \text{ for all } t, s \in I\},$$

then $Gu \in X$ for all $u \in \hat{X}$.

Claim 1. $G : X \rightarrow X$ is nondecreasing.

Let $u, v \in X$ be such that $u(t) \leq v(t)$ for all $t \in I$. By condition (F3) we have that $f_u \leq f_v$ on $[x_0, x_0 + R]$ and then from the definition of G it follows that

$$Gu(t) \leq Gv(t) \quad \text{for all } t \in I.$$

Claim 2. X is a complete lattice.

Let $\emptyset \neq Y \subset X$. We shall show only the existence of $\sup Y$, because the existence of $\inf Y$ is proved by a similar argument. We define

$$u^*(t) := \sup\{u(t) : u \in Y\} \quad \text{for all } t \in I.$$

Clearly $u^*(0) = x_0$ and $u^*(t)$ is finite for all $t \in I$. Now fix $s, t \in I$ and $u \in Y$. Then

$$u(s) \leq |u(s) - u(t)| + u(t) \leq L|t - s| + u^*(t).$$

Taking the supremum on the left-hand side we obtain that

$$u^*(s) \leq L|t - s| + u^*(t).$$

Interchanging s and t we have

$$u^*(t) \leq L|t - s| + u^*(s),$$

and combining both results

$$|u^*(s) - u^*(t)| \leq L|t - s|.$$

Therefore $u^* \in X$ and obviously $u^* = \sup Y$.

Claim 3. If $x \in X$ is a fixed point of G then x is an increasing solution of problem (1.1).

If $x = Gx$ then x is a solution of problem (4.5) with $u = x$. Moreover, since x is increasing and f_u is a selection of $F(\hat{u}(\cdot), \cdot)$ on $[x_0, x_0 + R]$ we have that x is also a solution of problem (1.1).

Conclusion. From claims 1 and 2 we can apply Tarski's fixed point theorem (see [12, Theorem 11.E]) to ensure the existence of (at least) one fixed point x of G . Then by claim 3 x is an increasing solution of (1.1). \square

REMARK 4.1 *In the case $\text{sgn}(x_1) = -1$ we can prove an analogous result on existence of decreasing solutions.*

Now we point out some sufficient conditions for having (F1) and (F3):

DEFINITION 4.1 *We say that a multivalued mapping $F : I \times J \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ is **strongly monotone nonincreasing with respect to its first variable** if for $s, t \in I$, $s \leq t$, we have*

$$\sup F(t, x) \leq \inf F(s, x) \quad \text{for all } x \in J.$$

PROPOSITION 4.1 *Let $F : I \times [x_0, x_0 + R] \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ be strongly monotone nonincreasing with respect to its first variable and let*

$$\sigma_F(x) := \sup F(T, x) \quad \text{and} \quad \iota_F(x) := \inf F(0, x) \quad \text{for all } x \in [x_0, x_0 + R].$$

Assume that σ_F and ι_F satisfy the following conditions:

- (a) $\iota_F, \sigma_F \in L^1(x_0, x_0 + R)$.
- (b) $x_1^2 + 2 \int_{x_0}^x \sigma_F(r) dr > 0$ for a.a. $x \in [x_0, x_0 + R]$.
- (c) $\frac{\max\{1, \iota_F, |\sigma_F|\}}{\sqrt{x_1^2 + 2 \int_{x_0}^x \sigma_F(r) dr}} \in L^1(x_0, x_0 + R)$.
- (d) $\int_{x_0}^{x_0+R} \frac{dx}{\sqrt{x_1^2 + 2 \int_{x_0}^x \iota_F(r) dr}} \geq T$.

If, moreover, for each $u \in \hat{X}$ the mapping $F_u = F(\hat{u}(\cdot), \cdot)$ has a measurable selection $f_u : [x_0, x_0 + R] \rightarrow \mathbb{R}$, then F satisfies (F1) and (F3).

Proof. For each $u \in \hat{X}$ let f_u be a measurable selection of $F_u|_{[x_0, x_0+R]}$. Let us see that f_u is an admissible selection on the right of x_0 : first, since $0 \leq \hat{u}(x) \leq T$ for all $t \in I$, we have

$$\sigma_F(x) \leq \inf F(\hat{u}(x), x) \leq \sup F(\hat{u}(x), x) \leq \iota_F(x),$$

because F is strongly nonincreasing in t . Hence

$$\sigma_F(x) \leq f_u(x) \leq \iota_F(x) \quad \text{for all } x \in [x_0, x_0 + R],$$

and, since f_u is measurable, condition (a) implies that $f_u \in L^1(x_0, x_0 + R)$. The remaining conditions of admissible selection on the right of x_0 admit analogous proofs.

Let us see that (F3) also holds: if $u, v \in \hat{X}$ are such that $u \leq v$ on I , then $\hat{u}(x) \geq \hat{v}(x)$ for all $x \in [x_0, x_0 + R]$. Since F is strongly nonincreasing in t , we have

$$\sup F(\hat{u}(x), x) \leq \inf F(\hat{v}(x), x) \quad \text{for all } x \in [x_0, x_0 + R].$$

Now if f_u is an admissible selection of F_u on $[x_0, x_0+R]$ and f_v is a corresponding one for F_v , the previous relation implies that $f_u \leq f_v$ on $[x_0, x_0 + R]$. \square

5 The particular case of differential equations

Plainly, theorem 4.1 covers the case of differential equations. It suffices to consider equations as inclusions with singleton-valued mappings. To the best of our knowledge, the application of theorem 4.1 to equations yields a new existence result, and that is why we think that it deserves to be stated clearly in a separate section of this paper.

Consider the initial value problem

$$\begin{cases} x''(t) = f(t, x(t)) & \text{for a.a. } t \in I := [0, T], \\ x(0) = x_0, \quad x'(0) = x_1 \geq 0. \end{cases} \quad (5.1)$$

THEOREM 5.1 *Assume that there exists $R > 0$ such that the function $f : I \times [x_0, x_0 + R] \rightarrow \mathbb{R}$ satisfies the following conditions:*

(I) *For each $u \in \hat{X}$ the mapping $f_u = f(\hat{u}(\cdot), \cdot)$ satisfies conditions (i) – (iv) in definition 3.1.*

(II) *$f(\cdot, x)$ is monotone nonincreasing for each $x \in [x_0, x_0 + R]$.*

(III) *There exists $M \in L^1(x_0, x_0 + R)$ such that for all $(t, x) \in I \times [x_0, x_0 + R]$ we have*

$$f(t, x) \leq M(x).$$

Then (5.1) has an increasing solution.

As consequence of Theorem 5.1 we have the following useful corollary.

COROLLARY 5.1 *Assume that there exists $R > 0$ such that the function $f : I \times [x_0, x_0 + R] \rightarrow \mathbb{R}$ satisfies the following conditions:*

1. *For each $u \in \hat{X}$ the mapping $f_u(\cdot) = f(\hat{u}(\cdot), \cdot)$ is measurable on $[x_0, x_0 + R]$.*

2. *$f(\cdot, x)$ is monotone nonincreasing for each $x \in [x_0, x_0 + R]$.*

3. *There exist $m, M \in L^1(x_0, x_0 + R)$ such that for all $(t, x) \in I \times [x_0, x_0 + R]$ we have*

$$0 < m(x) \leq f(t, x) \leq M(x).$$

4. $\frac{\max\{1, M(\cdot)\}}{\sqrt{x_1^2 + 2 \int_{x_0}^{\cdot} m(r) dr}} \in L^1(x_0, x_0 + R)$.
5. $\int_{x_0}^{x_0+R} \frac{1}{\sqrt{x_1^2 + 2 \int_{x_0}^x M(r) dr}} dx \geq T$.

Then (5.1) has an increasing solution.

Proof. In view of Theorem 5.1 we only have to prove that our assumptions imply that for each $u \in \hat{X}$ the function $f_u(\cdot) = f(\hat{u}(\cdot), \cdot)$ satisfies conditions (i) – (iv) in definition (3.1). From conditions 1 and 3 it follows that $f_u \in L^1(x_0, x_0 + R)$. Furthermore conditions (ii), (iii) and (iv) in definition (3.1) are easily deduced from 3, 4 and 5. \square

REMARK 5.1 *Concerning conditions 4 and 5 in corollary 5.1, we remark that when m and M are constants then condition 4 is always satisfied and moreover*

$$\int_{x_0}^{x_0+R} \frac{1}{\sqrt{x_1^2 + 2M(x - x_0)}} dx = \frac{1}{M} \left(\sqrt{x_1^2 + 2MR} - x_1 \right).$$

Therefore condition 5 can be expressed simply as

$$R \geq x_1 T + \frac{MT^2}{2}.$$

REMARK 5.2 *It is well-known that if $f : I \times [x_0, x_0 + R] \rightarrow \mathbb{R}$ satisfies the following “reversed” Carathéodory conditions,*

(C1) *for a.a. $x \in [x_0, x_0 + R]$ the function $f(\cdot, x)$ is continuous on I ,*

(C2) *for each $t \in I$ the function $f(t, \cdot)$ is measurable on $[x_0, x_0 + R]$.*

then $f(v(\cdot), \cdot)$ is measurable on $[x_0, x_0 + R]$ whenever $v : [x_0, x_0 + R] \rightarrow I$ is measurable. In particular if f satisfies (C1) and (C2) then f satisfies the condition 1 of Corollary 5.1.

EXAMPLE 5.1 *Consider the problem*

$$\begin{cases} x''(t) = g(t) + h(x(t)) & \text{for a.a. } t \in [0, \frac{1}{2}], \\ x(0) = 0, \quad x'(0) = 0, \end{cases} \quad (5.2)$$

where $g : [0, \frac{1}{2}] \rightarrow \mathbb{R}$ is monotone nonincreasing,

$$0 \leq g(t) \leq 1 \quad \text{for all } t \in [0, \frac{1}{2}],$$

and $h : [0, \frac{1}{2}] \rightarrow \mathbb{R}$ is given by

$$h(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \in (0, \frac{1}{2}], \\ 1 & \text{if } x = 0. \end{cases}$$

It is easy to check that the assumptions of Corollary 5.1 hold with $f(t, x) = g(t) + h(x)$ and therefore the problem 5.2 has an increasing solution on $[0, \frac{1}{2}]$.

We point out that another recent results such as [2, Theorem 1.1] and [6, Corollary 3.1] are not applicable to our example 5.1.

Finally, note that f tends to $+\infty$ as x tends to 0, hence the equation is singular at the initial condition.

As an immediate consequence of corollary 5.1 we have an existence result for the boundary value problem

$$x'' = f(t, x), \quad x(-T) = x(T), \quad x'(-T) = -x'(T). \quad (5.3)$$

COROLLARY 5.2 *Suppose that $f : [-T, T] \times [x_0, x_0 + R] \rightarrow \mathbb{R}$ satisfies conditions 1 – 5 in corollary 5.1 on the set $[0, T] \times [x_0, x_0 + R]$ with $x_1 = 0$.*

Suppose moreover that for each $(t, x) \in [-T, T] \times [x_0, x_0 + R]$ we have $f(-t, x) = f(t, x)$.

Then the problem (5.3) has an even solution which increases on $[0, T]$.

Proof. The assumptions and corollary 5.1 imply that

$$x'' = f(t, x), \quad x(0) = x_0, \quad x'(0) = 0,$$

has an increasing solution $x : [0, T] \rightarrow \mathbb{R}$. Now since $x'(0) = 0$ and f is even with respect to t we conclude that a solution of (5.3) in the conditions of the statement is given by

$$v(t) = \begin{cases} x(-t), & \text{if } t \in [-T, 0), \\ x(t), & \text{if } t \in [0, T]. \end{cases}$$

In a similar way we can have an existence result of odd solutions for

$$x'' = f(t, x), \quad x(-T) = -x(T), \quad x'(-T) = x'(T).$$

□

References

- [1] D.C. Biles, A necessary and sufficient condition for existence of solutions for differential inclusions, *Nonlinear Anal.* **31** (1998), 3-4, 311-315.
- [2] D.C. Biles and E. Schechter, Solvability of a finite or infinite system of discontinuous quasimonotone differential equations, *Proc. Amer. Math. Soc.* **128**, 5 (2000), 3349-3360.
- [3] Y. K. Chang and W. T. Li, Existence results for second order impulsive functional differential inclusions, *J. Math. Anal. Appl.* **301** (2005), 477-490.
- [4] J.Á. Cid, Existence of positive solutions for second-order ODEs with reverse Carathéodory functions, *Nonlinear Anal.* **54** (2003), 109-122.
- [5] P. Douka and N. S. Papageorgiou, Extremal solutions for nonlinear second order differential inclusions, *Math. Nachr.* **278** (2005), no. 1-2, 43-52.
- [6] S. Heikkilä and S. Seikkala, On the existence of extremal solutions of phi-Laplacian initial and boundary value problems, *Int. J. Pure Appl. Math.* **17** (2004), no. 1, 119-138.
- [7] R. López Pouso, A new approximation scheme for first-order ordinary differential equations with non-negative right-hand sides, *Nonlinear Anal.* **54** (2003), 361-372.
- [8] R. López Pouso, Necessary and sufficient conditions for existence and uniqueness of solutions for second order autonomous differential equations, *J. London Math. Soc.* (2) **71** (2005), 397-414.

- [9] E. J. McShane, *Integration*, Princeton University Press, Princeton, 1967 (Seventh edition. First edition: Princeton University Press, 1944).
- [10] D. Repovš and P. V. Semenov, *Continuous selections of multivalued mappings*, Kluwer Academic Publishers, Dordrecht, 1998.
- [11] D.H. Wagner, Survey of measurable selection theorems, *SIAM J. Control and Optimization*, **15** (1977), 5, 859-903.
- [12] Zeidler, E., *Nonlinear functional analysis and its applications. I. Fixed-point theorems*, Springer-Verlag, New York (1986).