AN EXCURSION THROUGH THE DOUBLE SIDEDNESS OF THE MATRIX INVERSE

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Abstract. We identify a gap between the existence of a left-inverse and the existence of the inverse of a matrix. By filling that gap in a new way we provide a novel proof of the two sidedness of the matrix inverse.

We propose a trip into linear algebra by discussing the reason for a well-known elementary fact: the two sidedness of the inverse of a matrix.

Theorem 1. $A \cdot B = I_n \Rightarrow B \cdot A = I_n$.

Throughout this paper A and B always shall denote $n \times n$ matrices over a field K, I_n shall stand for the identity matrix of order n and 0_n for the zero column vector in K^n. For a given matrix A we shall denote its columns by A^i, $i = 1, \ldots n$.

The key for understanding Theorem 1 is summarized in the following result.

Theorem 2. *Suppose that $A \cdot B = I_n$, then:*

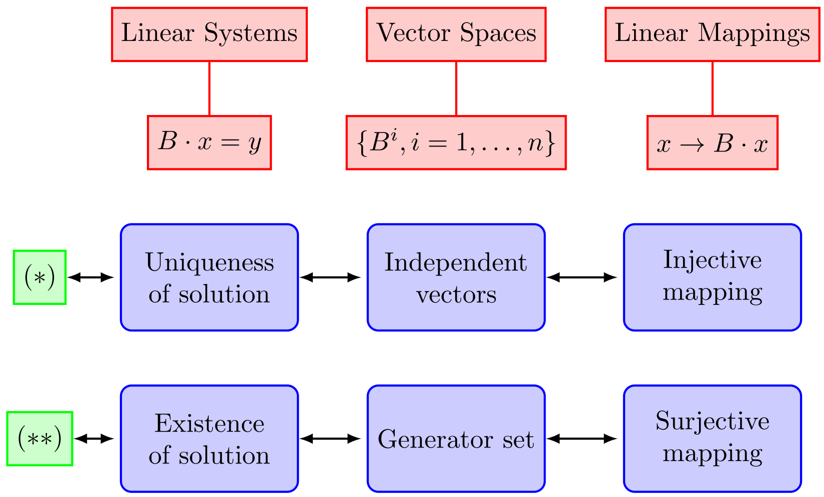
(i) $\boxed{B \cdot x = 0_n \Rightarrow x = 0_n. \quad (*)}$
(ii) $\boxed{\forall\, y \in K^n\, \exists\, x \in K^n \,/\, B \cdot x = y \quad (**)} \Rightarrow B \cdot A = I_n$.

Proof. (i) $B \cdot x = 0_n \Rightarrow A \cdot (B \cdot x) = A \cdot 0_n \Rightarrow (A \cdot B) \cdot x = 0_n \Rightarrow I_n \cdot x = 0_n \Rightarrow x = 0_n$.
(ii) From $(**)$ it follows that for I_n^i there exists $C^i \in K^n$ such that $B \cdot C^i = I_n^i$, for each $i = 1, \ldots n$, and then $B \cdot C = I_n$. Then, it suffices to show that

$$A = A \cdot I_n = A \cdot (B \cdot C) = (A \cdot B) \cdot C = I_n \cdot C = C.$$

□

Both properties $(*)$ and $(**)$ are ubiquitous and more than familiar to any algebra freshman. Their well-known meaning in different frameworks are indicated in the following diagram that summarizes part of the information given in [6, Chapter 2, Theorem 8]



Therefore, if we have $A \cdot B = I_n$ and in order to prove that also $B \cdot A = I_n$

what we get is $(*)$ *but what we need is* $(**)$.

So, any proof of "$(*) \Rightarrow (**)$" will provide a proof of Theorem 1: for instance, the proof by Fearnley-Sander contained in [3] and the one by Paparella in [8] would fit this approach. Also the elementary fact that an underdetermined homogeneous linear system has a nontrivial solution, a result derived from the row echelon form of the coefficient matrix, leads to the following proof of Theorem 1 that is new to the best of our knowledge (see [9] for a different proof based on the same principle and [1, 2, 3, 4, 5, 7] for other approaches).

Proof of Theorem 1. By Theorem 2 it is enough to prove that $(*)$ implies $(**)$. Now, for each $y \in K^n$ the homogeneous system

$$\left(\; B^1 \mid B^2 \mid \ldots \mid B^n \mid y \; \right) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} = 0_n,$$

has n equations and $n+1$ unknowns and therefore it has a nontrivial solution. From $(*)$ it follows that $x_{n+1} \neq 0$ and then

$$B \cdot \begin{pmatrix} -x_1/x_{n+1} \\ -x_2/x_{n+1} \\ \vdots \\ -x_n/x_{n+1} \end{pmatrix} = y,$$

so $(**)$ holds. □

Note that, Theorem 1 holds due to the fortunate fact that in finite-dimensional vector spaces injective and surjective mappings coincide! However, as it is well know, this is not true when considering infinite-dimensional vector spaces and, in fact, Theorem 1 fails in this setting as the counterexample pointed out in [1, Section 4] shows.

We hope this short excursion had shed some light into the fascinating landscape of linear algebra and the reader could find something valuable in the trip.

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