A simple method to find out when an ordinary differential equation is separable

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We present an alternative method to that of [5] to teach the students how to discover if a differential equation $y' = f(x, y)$ is separable or not when the nonlinearity $f(x, y)$ is not explicitly factorized.

Our approach is completely elementary and provides a necessary and sufficient condition for separability, as well as simple formulas for the explicit factors of $f(x, y)$.

Keywords: Separation of variables; separable functions; first order differential equations.

1 Introduction

A first order differential equation

$$y'(x) = f(x, y), \tag{1.1}$$

is called separable if the right-hand side can be factorized as $f(x, y) = \phi(x)\psi(y)$. All introductory textbooks on differential equations show the beginners how to solve formally these kind of equations by separating the variables and integrating both sides of the equation, which leads us to the implicit solution formula

$$\int \frac{dy}{\psi(y)} = \int \phi(x)dx. \tag{1.2}$$

We recommend [4] for an interesting discussion about some properties of the solutions that can be obtained by the separation of variables method.

However, the students often fail to recognize a "hidden" separable equation such as for example the given by

$$f(x, y) = \mathrm{e}^{x^2+y^2}(\cos(x + y) + \cos(x - y)).$$

Contrary to the exact equations for which the criterium of exactness is standard, the following test for separability due to Scott [5] is much less known (among the usual textbooks on differential equations we have only been able to find it in [1, 2]).

THEOREM 1.1. *Let $D \subset \mathbb{R}^2$ be and open convex set and $f : D \to \mathbb{R}$.*

(1) If there are differentiable functions $\phi(x)$ and $\psi(y)$ such that $f(x,y) = \phi(x)\psi(y)$ for all $(x,y) \in D$, then

$$f(x,y)f_{xy}(x,y) = f_x(x,y)f_y(x,y) \quad \text{for all } (x,y) \in D.$$

(2) If f, f_x, f_y and f_{xy} are continuous in D, $f(x,y)$ is never 0 in D and

$$f(x,y)f_{xy}(x,y) = f_x(x,y)f_y(x,y) \quad \text{for all } (x,y) \in D,$$

then there are continuously differentiable functions $\phi(x)$ and $\psi(y)$ such that $f(x,y) = \phi(x)\psi(y)$ for all $(x,y) \in D$.

Moreover, in case (2) the formulas for $\phi(x)$ and $\psi(y)$ given in the proof of [5, Proposition 2] are the following:

$$\phi(x) = \exp(\beta(x)) \quad \text{and} \quad \psi(y) = \exp(\gamma(y)), \tag{1.3}$$

where $\beta(x) = \int \alpha(x)dx$, $\alpha(x) = \frac{\partial}{\partial x}\ln(f(x,y))$ and $\gamma(y) = \ln(f(x,y)) - \beta(x)$ (since $f(x,y) \neq 0$ we consider here that $f(x,y) > 0$).

A serious inconvenient of Theorem 1.1 is that the function $f(x,y)$ must be nonzero on D in order to obtain the validity of part (2), as the following counterexample shows (see [5])

$$f(x,y) = \begin{cases} x^2 \mathrm{e}^y, & \text{if } x \geq 0, \\ x^2 \mathrm{e}^{2y}, & \text{if } x \leq 0. \end{cases}$$

The condition that f is nonzero on D can be dropped but assuming that f is an analytic function of both variables x and y. However, we notice that by Peano's theorem the differential equation (1.1) is always solvable for every continuous f and therefore the analyticity of f is a too strong condition from the viewpoint of the theory of differential equations (for example it does not cover standard examples such as $y' = \sqrt{|y|}$).

In our opinion Theorem 1.1 and the formulas (1.3) are unnecessarily complicated for students in order to characterize a separable equation. The main goal of this note is to present a simpler condition which is both necessary and sufficient for a differential equation to be separable.

2 A simple criterium for separability

Next we present our main result.

THEOREM 2.1. *Let D a subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$. Then, $f(x,y) = \phi(x)\psi(y)$ for all $(x,y) \in D$ if and only if*

$$f(a,b)f(x,y) = f(x,b)f(a,y) \quad \text{for all } (a,b),\ (x,y) \in D \tag{2.4}$$

Proof. If $f(x,y) = \phi(x)\psi(y)$ then clearly condition (2.4) holds.

To prove the reciprocal implication assume that $f(x,y)$ is not identically zero on D (in other case $f \equiv 0$ is already separable). Let $(a_0,b_0) \in D$ be such that $f(a_0,b_0) \neq 0$ and define $\phi(x) = \frac{f(x,b_0)}{f(a_0,b_0)}$ and $\psi(y) = f(a_0,y)$. Therefore, from (2.4) it follows that

$$\phi(x)\psi(y) = \frac{f(x,b_0)f(a_0,y)}{f(a_0,b_0)} = \frac{f(a_0,b_0)f(x,y)}{f(a_0,b_0)} = f(x,y).$$

□

REMARK 2.1. *(1).- It is clear that Theorem 2.1 is applicable to a wider class of functions than Theorem 1.1, because we don't impose regularity assumptions on f and moreover f is allowed to vanish.*

(2).- Theorem 1.1 has been generalized for functions of several variables in [3]. Theorem 2.1 can also be easily generalized for functions of n variables. Indeed, it is satisfied that $f(x_1,x_2,\dots,x_n) = h_1(x_1)h_2(x_2)\dots h_n(x_n)$ if and only if

$$f(a_1,a_2,\dots,a_n)^{n-1}f(x_1,x_2,\dots,x_n) = f(x_1,a_2,\dots,a_n)\cdot\ldots\cdot f(a_1,a_2,\dots,x_n),$$

for each $(a_1,a_2,\dots,a_n),(x_1,x_2,\dots,x_n) \in \mathbb{R}^n$.

Observing the proof of Theorem 2.1 we arrive at the following consequence whenever f is not identically zero and which provides explicitly the factors ϕ and ψ needed in order to construct the solution formula (1.2).

COROLLARY 2.1. *Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and assume that there exists $(x_0,y_0) \in D$ such that $f(x_0,y_0) \neq 0$.*
Then, $f(x,y) = \phi(x)\psi(y)$ for all $(x,y) \in D$ if and only if

$$f(x,y) = \frac{f(x,y_0)f(x_0,y)}{f(x_0,y_0)} \quad \text{for all } (x,y) \in D. \tag{2.5}$$

Moreover, if condition (2.5) holds we can choose $\phi(x) = \frac{f(x,y_0)}{f(x_0,y_0)}$ and $\psi(y) = f(x_0,y)$ as factors of $f(x,y)$.

The information of Corollary 2.1 can be summarize in the following algorithm:

Test for separability of $y'(x) = f(x, y)$

(1) Choose any (x_0, y_0) such that $f(x_0, y_0) \neq 0$.

(2) Construct the functions $\phi(x) = \frac{f(x,y_0)}{f(x_0,y_0)}$ and $\psi(y) = f(x_0, y)$.

(3) The equation is separable if and only if $f(x, y) = \phi(x)\psi(y)$.

Finally we are going to check the applicability of the preceding test with a couple of examples yet mentioned trough the text.

Example 2.1. Consider the function

$$f(x, y) = \mathrm{e}^{x^2+y^2}(\cos(x + y) + \cos(x - y)),$$

presented at the beginning of the paper. If we choose $(x_0, y_0) = (0, 0)$, for which $f(x_0, y_0) = 2$, then

$$\phi(x) = \frac{f(x, 0)}{f(0, 0)} = \mathrm{e}^{x^2}\cos(x) \quad \text{and} \quad \psi(y) = f(0, y) = 2\mathrm{e}^{y^2}\cos(y).$$

Now it is an easy calculation with the trigonometric identities to verify that indeed $f(x, y) = \phi(x)\psi(y)$.

Example 2.2. Consider now the function

$$f(x, y) = \begin{cases} x^2\mathrm{e}^y, & \text{if } x \geq 0, \\ x^2\mathrm{e}^{2y}, & \text{if } x \leq 0, \end{cases}$$

which is the counterexample of Scott (see [5]). If we choose $(x_0, y_0) = (1, 0)$, for which $f(x_0, y_0) = 1$, then

$$\phi(x) = \frac{f(x, 0)}{f(1, 0)} = x^2 \quad \text{and} \quad \psi(y) = f(1, y) = \mathrm{e}^y.$$

It is clear that $f(x, y) \neq \phi(x)\psi(y)$ so $f(x, y)$ is not separable.

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