Research Article

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On the existence of periodic oscillations for pendulum-type equations

Abstract: We provide new sufficient conditions for the existence of T-periodic solutions for the ϕ-laplacian pendulum equation $\left(\phi(x')\right)' + k\,x' + a\sin x = e(t)$, where $e \in \widetilde{C}_T$. Our main tool is a continuation theorem due to Capietto, Mawhin and Zanolin and we improve or complement previous results in the literature obtained in the framework of the classical, the relativistic and the curvature pendulum equations.

Keywords: Periodic solution, ϕ-Laplacian, pendulum equation, relativistic pendulum, continuation theorem

MSC: 34C25

1 Introduction

Let us consider the classical pendulum equation

$$x'' + k\,x' + a\sin x = e(t), \tag{1.1}$$

where

(H0) $k \geq 0$, $a > 0$, $T > 0$ and $e \in \widetilde{C}_T$ the set of continuous T-periodic functions with mean value $\bar{e} :=$
$$\frac{1}{T}\int_0^T e(t)dt = 0.$$

The search for T-periodic solutions of (1.1) has been a fruitful subject over the last century, see [12, 13], and it is well-known the following general solvability result: there exist $s_- = s_-(k, a, e)$ and $s_+ = s_+(k, a, e)$ with

$$-a \leq s_- \leq s^+ \leq a,$$

such that the equation

$$x'' + k\,x' + a\sin x = e(t) + s, \tag{1.2}$$

has a T-periodic solution if and only if $s \in [s_-, s_+]$. Moreover, whenever $s \in\ ]s_-, s_+[$ there exist at least two geometrically different solutions of (1.2). Surprisingly, it is still an open problem to know if the degeneracy condition $s_- = s_+$ can be attained or not.

Following the preceding notation, (1.1) has a T-periodic solution if and only if $0 \in [s_-, s_+]$. This is always true in the conservative framework, i.e. when $k = 0$, as it was proven by Hamel in [9] and later improved in [16] by adding a second geometrically different T-periodic solution.

However, the solvability of (1.1) it is not longer ensured in the presence of a friction term: indeed, it has been proved in [19] that for each k, a, $T > 0$ there exists $e \in \widetilde{C}_T$ such that (1.1) has not T-periodic solutions. Of course, sufficient conditions for the existence of T-periodic solutions for (1.1) are known, such as

$$\sqrt{T}\|e\|_2 < k\sqrt{\pi}\sqrt{3}, \quad ([16,\ \text{Remark 1}]),$$

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$$\|e\|_\infty < a, \quad ([16, \text{Remark 2}]),$$

or

$$T(2T + k\pi + \delta E) < 4, \quad ([6, \text{Remark 4}]),$$

where $\delta E = \max\limits_{t\in[0,T]} \int\limits_0^t e(s)ds - \min\limits_{t\in[0,T]} \int\limits_0^t e(s)ds.$

Recently, Brézis and Mawhin, [3, Corollary 1], have proven that the relativistic conservative pendulum, that is

$$\left(\frac{x'}{\sqrt{1 - \frac{x'^2}{c^2}}}\right)' + a \sin x = e(t), \tag{1.3}$$

where $c > 0$ is the speed of the light in the vacuum, $a > 0$ and $e \in \widetilde{C}_T$, always has a T-periodic solution. Later, Bereanu and Torres, [2, Corollary 1.2], added the existence of a second T-periodic solution. So, in spite of the technical difficulties the solvability issue for (1.3) is analogous to the classical setting. To the contrary, Torres proved in [20] that the forced relativistic pendulum

$$\left(\frac{x'}{\sqrt{1 - \frac{x'^2}{c^2}}}\right)' + k x' + a \sin x = e(t), \tag{1.4}$$

always has a T-periodic solution for all k, a, $T > 0$ and $e \in \widetilde{C}_T$ provided that

$$2cT < 1.$$

That condition has been improved later in [21, Corollary 3] (see also [7, Remark 1]) and the best bound until now for the right-hand side, up to our knowledge, was obtained in [1, Theorem 1], namely

$$2cT < 2\sqrt{3}\pi. \tag{1.5}$$

However it is not know if there exist examples of non-continuation of periodic oscillations for (1.4) for bigger values of the period T.

Recently, it has been proven in [11, Theorem 2.1] the solvability of (1.4) under the following alternative condition to (1.5),

$$2c_\star T < 2\pi, \tag{1.6}$$

where the constant $c_\star = c_\star(k, a, T, \|e\|_\infty)$ is implicitly defined by the equation

$$c_\star = \frac{c\left(kTc_\star + 3k\pi + 2(a + \|e\|_\infty)T\right)}{\sqrt{c^2 + \left(kTc_\star + 3k\pi + 2(a + \|e\|_\infty)T\right)^2}}. \tag{1.7}$$

Notice that $0 < c_\star < c$ but that neither (1.5) nor (1.6) are implied by each other, so they are independent.

Both equations, (1.1) and (1.4), fit into the so-called ϕ-Laplacian equations

$$\left(\phi(x')\right)' + k x' + a \sin x = e(t), \tag{1.8}$$

where

$(H1)\phi :] - A, A[\longrightarrow] - B, B[$ is an increasing and odd homeomorphism with $0 < A, B \le +\infty$.

Note that in the classical pendulum equation (1.1) we have $\phi(z) = z$ while in the relativistic pendulum (1.4) is $\phi(z) = \dfrac{z}{\sqrt{1 - \frac{z^2}{c^2}}}$. Notice also that other important homeomorphisms, like the p-laplacian $\phi(z) = z|z|^{p-2}$, with $p \ge 2$, and the mean curvature operator $\phi(z) = \dfrac{z}{\sqrt{1 + z^2}}$ also satisfy $(H1)$.

The present paper is organized as follows: in next section we present our main result, several consequences and we discuss their relevance with the literature. In Section 3 we collect the auxiliary results that we will need for the proof of our main result that is postponed until Section 4. Our main tool will be a Capietto-Mawhin-Zanolin continuation theorem given in [4]. Finally, in Section 5 we point out a striking difference on the dynamic behaviour between both the classical and relativistic pendulums and the "curvature" pendulum.

2 Main results

The following is the main result in this paper: a sufficient condition for the existence of multiple T-periodic solutions for equation (1.8). In the particular case of the relativistic pendulum equation (1.4) we improve simultaneously both conditions (1.5) and (1.6), see Corollary 2. It also provides an apparently new solvability condition even for the classical pendulum equation (1.1), see Theorem 6.

Theorem 1. *Assume (H0), (H1) and moreover*

$$2k\pi + \frac{aT}{2} + \frac{\|e\|_{L^1}}{2} < B, \tag{2.1}$$

and

$$\frac{T}{2}\phi^{-1}\left(2k\pi + \frac{aT}{2} + \frac{\|e\|_{L^1}}{2}\right) < \pi. \tag{2.2}$$

Then there exist at least two geometrically different T-periodic solutions of (1.8), x_1 and x_2, such that:

$$-\pi < x_1(t) < \pi \text{ for all } t \in \mathbb{R} \text{ and } x_1(t_1) = 0 \text{ for some } t_1 \in [0, T], \tag{2.3}$$

$$0 < x_2(t) < 2\pi \text{ for all } t \in \mathbb{R} \text{ and } x_2(t_2) = \pi \text{ for some } t_2 \in [0, T]. \tag{2.4}$$

We remark that condition (2.1) is implicitly assumed in (2.2) and it is trivially fulfilled for unbounded operators (that is, when $B = +\infty$).

2.1 The relativistic pendulum

Taking $\phi(z) = \dfrac{z}{\sqrt{1 - \frac{z^2}{c^2}}}$ in Theorem 1 we obtain the following existence result for equation (1.4).

Theorem 2. *Let us suppose (H0) and*

$$2T\hat{c} < 4\pi, \tag{2.5}$$

where

$$\hat{c} = \frac{c\left(2k\pi + \frac{aT}{2} + \frac{\|e\|_{L^1}}{2}\right)}{\sqrt{c^2 + \left(2k\pi + \frac{aT}{2} + \frac{\|e\|_{L^1}}{2}\right)^2}}. \tag{2.6}$$

Then equation (1.4) has at least two geometrically different T-periodic solutions satisfying (2.3) and (2.4).

Remark 3. As we have previously noticed condition (2.5) improves simultaneously both conditions (1.5) and (1.6) since $0 < \hat{c} < c^* < c$.

Example 4. For each $\epsilon > 0$ let us consider the relativistic pendulum equation

$$\left(\frac{x'}{\sqrt{1 - x'^2}}\right)' + \epsilon^2 x' + \epsilon^3 \sin x = \epsilon^3 \sin \epsilon t, \tag{2.7}$$

where we have normalized $c = 1$.

Notice that $T = \dfrac{2\pi}{\epsilon}$ and $\lim_{\epsilon \to 0^+} T = +\infty$, so condition (1.5) is not satisfied for small enough $\epsilon > 0$. On the other hand, since

$$\hat{c} = \frac{(2 + 3\pi)\epsilon^2}{\sqrt{1 + (2 + 3\pi)^2 \epsilon^4}}$$

then Theorem 2 provides two T-periodic solutions of (2.7) for each $0 < \epsilon < 0.0878689$.

2.2 The *p*-laplacian pendulum

Taking now $\phi(z) = |z|^{p-2}z$ for $p \geq 2$ in Theorem 1 we obtain the following existence result for the *p*-laplacian pendulum. This equation seems to be skipped in the literature and only few references explicitly deal with it, see [15].

Theorem 5. *Suppose (HO), $p \geq 2$ and*

$$\frac{T}{2}\left(2k\pi + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2}\right)^{\frac{1}{p-1}} < \pi. \tag{2.8}$$

Then equation

$$(|x'|^{p-2}x')' + k\,x' + a\sin x = e(t), \tag{2.9}$$

has at least two geometrically different T-periodic solutions satisfying (2.3) *and* (2.4).

2.3 The classical pendulum

If in Theorem 5 we take $p = 2$ we obtain an existence result for (1.1) that it is new to the best of our knowledge.

Theorem 6. *Suppose (HO) and*

$$\frac{T}{2}\left(2k\pi + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2}\right) < \pi. \tag{2.10}$$

Then equation (1.1) *has at least two geometrically different T-periodic solutions satisfying* (2.3) *and* (2.4).

Clearly, if $\frac{\|e\|_{L^1}}{2} < \frac{2\pi}{T}$ then (2.10) is satisfied for small enough k, $a > 0$. This observation is applied in the following result.

Corollary 7. *Let us fix $T > 0$ and $e \in \widetilde{C}_T$ such that*

$$\frac{\|e\|_{L^1}}{2} < \frac{2\pi}{T}. \tag{2.11}$$

Then for any k, $a > 0$ such that

$$2k\pi + a\frac{T}{2} < \frac{2\pi}{T} - \frac{\|e\|_{L^1}}{2},$$

the equation (1.1) *has at least two geometrically different T-periodic solutions satisfying* (2.3) *and* (2.4).

2.4 The curvature pendulum

Taking now $\phi(z) = \dfrac{z}{\sqrt{1 + z^2}}$ in Theorem 1 we obtain the following existence result for the "curvature" pendulum. Different sufficient conditions for the solvability of curvature pendulum equations were given in [1, 17, 18].

Theorem 8. *Suppose (HO) and moreover*

$$2k\pi + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2} < 1 \tag{2.12}$$

and

$$\frac{T}{2}\frac{\left(2k\pi + \frac{aT}{2} + \frac{\|e\|_{L^1}}{2}\right)}{\sqrt{1 - \left(2k\pi + \frac{aT}{2} + \frac{\|e\|_{L^1}}{2}\right)^2}} < \pi. \tag{2.13}$$

Then equation

$$\left(\frac{x'}{\sqrt{1 + x'^2}}\right)' + k\,x' + a\sin x = e(t), \tag{2.14}$$

has at least two geometrically different T-periodic solutions satisfying (2.3) *and* (2.4).

3 Auxiliary results

By means of the change of variables $y = \phi(x') + kx$, to find a T-periodic solution of (1.8) is equivalent to solve the following periodic boundary value problem for a first order system

$$\begin{cases} x' = \phi^{-1}(y - kx), & x(0) = x(T), \\ y' = -a\sin(x) + e(t), & y(0) = y(T). \end{cases} \tag{3.1}$$

This changes of variables was introduced in [14], inspired by the Liénard plane, and used also in [11]. Notice that in case ϕ is a bounded homeomorphism (that is, $B < +\infty$), the right-hand side of system (3.1) is not longer defined on the whole plane $\mathbb{R}^2$. With this idea in mind let us consider the periodic BVP

$$z' = F(t, z), \quad z(0) = z(T), \tag{3.2}$$

assuming that

$$F(t, z) := f(t, z; 1),$$

where $f : [0, T] \times G \times [0, 1] \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$ is an open set and f is a Carathéodory function such that for $\lambda = 0$ the map f is autonomous, that is, there exists a continuous function $f_0 : G \to \mathbb{R}^n$ such that

$$f_0(z) := f(t, z; 0),$$

for a.a. $t \in [0, T]$ and all $z \in G$.

The following result is just a small modification of [4, Corollary 3] to deal with functions $f(t, \cdot, \lambda)$ not defined on the whole $\mathbb{R}^n$.

Lemma 9. *Let Ω be a bounded and open subset of $\mathbb{R}^n$ such that $\overline{\Omega} \subset G$ and suppose that the following conditions are satisfied:*
(CMZ1) ("Bound set" condition) For any $\lambda \in [0, 1)$ and any z solution of

$$z' = f(t, z; \lambda), \quad z(0) = z(T), \tag{3.3}$$

such that $z(t) \in \overline{\Omega}$ for all $t \in [0, T]$, it follows that $z(t) \in \Omega$ for all $t \in [0, T]$;
(CMZ2) $d_B(f_0, \Omega, 0) \neq 0$, where d_B stands for the usual Brouwer degree in $\mathbb{R}^n$.

Then, problem (3.2) has at least one solution $z(t)$ such that $z(t) \in \overline{\Omega}$ for all $t \in [0, T]$.

Proof. By the Tietze-Dugundji Theorem, see [22, Proposition 2.1], the function $f : [0, T] \times \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ admits a continuous extension $\tilde{f} : [0, T] \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$. Now, we can apply [4, Corollary 3] to obtain a solution $z(t)$ of problem (3.2) with $F(t, z) := \tilde{f}(t, z, 1)$ such that $z(t) \in \overline{\Omega}$. Then $z(t)$ is also a solution of (3.2) with $F(t, z) := f(t, z, 1)$. □

In order to apply Lemma 9 to problem (3.1) the following estimates about its possible solutions are essential.

Lemma 10. *Let us assume (H0) and (H1). For any $\lambda \in [0, 1]$, let (x, y) a solution of the problem*

$$\begin{cases} x' = \phi^{-1}(y - kx), & x(0) = x(T), \\ y' = -a\sin(x) + \lambda e(t), & y(0) = y(T). \end{cases} \tag{3.4}$$

such that

$$\|x - m\pi\|_\infty \leq l, \quad \text{for some } 0 < l < \pi \text{ and } m \in \mathbb{Z}.$$

Then, the following estimates hold:
(i) There exists $t_0 \in [0, T]$ such that $x(t_0) = m\pi$,
(ii) $\|y - km\pi\|_\infty \leq kl + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2}$,

(iii) $\|x - m\pi\|_\infty \leq \max\limits_{t \in [0,T]} x(t) - \min\limits_{t \in [0,T]} x(t) \leq \frac{T}{2}\phi^{-1}\left(2kl + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2}\right).$

Proof. Define the functions $\tilde{x} = x - m\pi$ and $\tilde{y} = y - km\pi$ which satisfy

$$\begin{cases} \tilde{x}' = \phi^{-1}(\tilde{y} - k\tilde{x}), & \tilde{x}(0) = \tilde{x}(T), \\ \tilde{y}' = -a\sin(\tilde{x} + m\pi) + \lambda e(t), & \tilde{y}(0) = \tilde{y}(T). \end{cases} \tag{3.5}$$

Integrating the second equation over a period we have

$$0 = \tilde{y}(T) - \tilde{y}(0) = \int_0^T \tilde{y}'(s)ds = -a\int_0^T \sin(\tilde{x}(s) + m\pi)ds,$$

where $-\pi < -l \leq \tilde{x}(t) \leq l < \pi$ for all $t \in [0, T]$. Since $a \neq 0$ it follows the existence of $t_0 \in [0, T]$ such that $\tilde{x}(t_0) = 0$ which is equivalent to (i).

Now, integrating the first equation over a period we get

$$0 = \tilde{x}(T) - \tilde{x}(0) = \int_0^T \tilde{x}'(s)ds = \int_0^T \phi^{-1}(\tilde{y}(s) - k\tilde{x}(s))ds,$$

and by (H1) it follows the existence of $t_1 \in [0, T]$ such that $\tilde{y}(t_1) = k\tilde{x}(t_1)$ and then

$$-k\pi < -kl \leq \tilde{y}(t_1) \leq kl < k\pi.$$

Extending $\tilde{y}$ periodically, if needed, there exists $t_2 \in \mathbb{R}$ such that $\tilde{y}(t_2) = \|\tilde{y}\|_\infty$ with $|t_2 - t_1| \leq \frac{T}{2}$. Without loss of generality let us suppose that $t_1 \leq t_2$. Using the second equation of (3.5) we obtain

$$\begin{aligned} \|\tilde{y}\|_\infty \quad &\leq \quad |\tilde{y}(t_1)| + \int_{t_1}^{t_2} |\tilde{y}'(s)|ds \\ &\leq \quad kl + \int_{t_1}^{t_2} a|\sin(\tilde{x}(s) + m\pi)|ds + \lambda\int_{t_1}^{t_2} e(s)^+ ds \\ &\leq \quad kl + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2}, \end{aligned}$$

and thus (ii) is proven (take into account that since $e \in \tilde{C}_T$ then $\int_0^T e^+(s)ds = \int_0^T e^-(s)ds$ and $\|e\|_{L^1} = 2\int_0^T e^+(s)ds$).

Finally, from (i), (ii) and the first equation of (3.5) it follows (iii). □

4 Proof of Theorem 1

By (2.2) there exists $l > 0$ such that

$$\frac{T}{2}\phi^{-1}\left(2k\pi + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2}\right) < l < \pi,$$

and let us consider the open bounded set in $\mathbb{R}^2$

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 : |x| < l,\ |y| < k\pi + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2}\}. \tag{4.1}$$

Now, our strategy to find a T-periodic solution x_1 satisfying (2.3) is to apply Lemma 9 to the homotopic system (3.4) and the set Ω_1. Notice that by (2.1) we have that $\overline{\Omega}_1 \subset G := \{(x, y) \in \mathbb{R}^2 : |y - kx| < B\}$. So it is enough to verity the following two claims.

Claim 1. For any $\lambda \in [0, 1]$ *and any* (x, y) *solution of* (3.4) *such that* $(x(t), y(t)) \in \overline{\Omega}_1$ *for all* $t \in [0, T]$, *it follows that* $(x(t), y(t)) \in \Omega_1$ *for all* $t \in [0, T]$.

Since $(x(t), y(t)) \in \overline{\Omega}_1$ for all $t \in [0, T]$ then (x, y) satisfies the assumptions in Lemma 10 with $m = 0$. Then, from estimates (iii) and (ii) we have

$$\|x\|_\infty \le \frac{T}{2} \phi^{-1} \left(2kl + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2} \right) < \frac{T}{2} \phi^{-1} \left(2k\pi + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2} \right) < l,$$

$$\|y\|_\infty \le kl + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2} < k\pi + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2}$$

and so $(x(t), y(t)) \in \Omega_1$ for all $t \in [0, T]$.

Claim 2. $d_B(f_0, \Omega_1, (0, 0)) \neq 0$ *where* $f_0(x, y) = (\phi^{-1}(y - kx), -a \sin(x))$.

Since $0 < l < \pi$ and $a \neq 0$, the only zero of f_0 in $\overline{\Omega}_1$ is $(0, 0)$ and then $d_B(f_0, \Omega_1, (0, 0))$ is well defined. Since Ω_1 is symmetric with $(0, 0) \in \Omega_1$ and f_0 is continuous in $\overline{\Omega}_1$ and odd, then by Borsuk's theorem it follows that $d_B(f_0, \Omega_1, (0, 0))$ is odd. Thus the claim follows.

Finally, the proof of the existence of a T-periodic solution x_2 satisfying (2.4) is analogous by using the open bounded set

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 : |x - \pi| < l, \ |y - k\pi| < k\pi + a\frac{T}{2} + \frac{\|e\|_{L^1}}{2}\}. \tag{4.2}$$

Indeed, the analogous to *Claim 1* follows from the estimates provided in Lemma 10 with $m = 1$. On the other hand, for the analogous to *Claim 2* consider the homeomorphism $T : \Omega_1 \to \Omega_2$ defined by $T(x, y) = (x + \pi, y + k\pi)$. By the Product Formula (see [8, Theorem 5.1]) we have

$$d_B(f_0, \Omega_2, (0, 0)) = d_B(f_0 \circ T, \Omega_1, (0, 0)) \cdot d_B(T, \Omega_1, z) = d_B(f_0 \circ T, \Omega_1, (0, 0)), \tag{4.3}$$

since $d_B(T, \Omega_1, z) = 1$ for any $z \in \Omega_2$. So we can apply again Borsuk's theorem to the right hand side of (4.3) since $(f_0 \circ T)(x, y) = (\phi^{-1}(y - kx), a \sin(x))$ is again a continuous and odd function.

Remark 11. Adding to (H1) the stronger regularity assumptions $\phi \in C^1(-\epsilon, \epsilon)$, with $\epsilon > 0$, and $\phi'(0) \neq 0$ we can compute exactly

$$d_B(f_0, \Omega_1, 0) = \text{sign}\left(J_{f_0}(0, 0)\right) = \text{sign}\left(-\frac{a}{\phi'(0)}\right) = -1.$$

5 A counterexample for the "curvature" pendulum

We have already stressed at Introduction that both equations (1.1), with $k = 0$, and (1.3) has a T periodic solution for any $a \in \mathbb{R}$ and $e \in \tilde{C}_T$. On the other hand, the "curvature" pendulum, also called the "sine-curvature" equation,

$$\left(\frac{x'}{\sqrt{1 + x'^2}} \right)' + a \sin x = e(t), \tag{5.1}$$

is a ϕ-laplacian type equation with a bounded operator, namely $\phi(z) = \frac{z}{\sqrt{1 - z^2}} \in (-1, 1)$. This fact leads to the following dramatic difference with respect to the classical/relativistic pendulum: for each $T > 0$ and $a \in \mathbb{R}$ there exists $e \in \tilde{C}_T$ such that equation (5.1) has not any solution defined on the whole interval $[0, T]$. In particular, for such $e \in \tilde{C}_T$ the equation (5.1) has not T periodic solutions in clear contrast with the classical and the relativistic settings. The key difference is that the kinetic energy of the "curvature" pendulum, that is $E_{kin}(x) := \frac{1}{\sqrt{1+x'^2}}$, is *a priori* bounded independently of the solution (quite surprisingly, even if the velocity
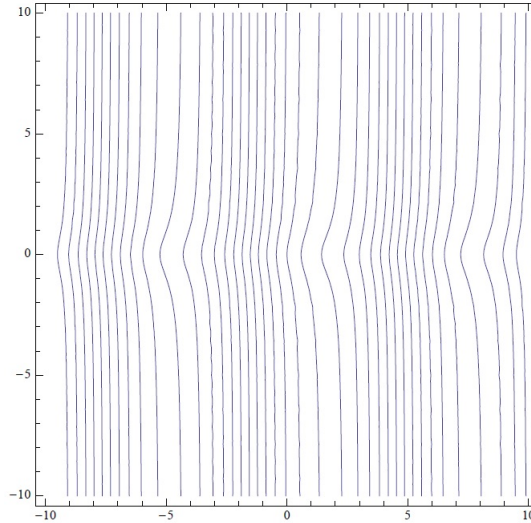
Fig. 1: Phase plane for (5.2) with the values $a = 1$ and $H = 1$.

of the solution tends to infinity its kinetic energy remains bounded!). The approach in this section is inspired on some previous ideas developed in [10] to obtain counterexamples to the existence of solutions between well-ordered lower and upper solutions without the so-called Nagumo condition.

Lemma 12. *Let us consider $a \in \mathbb{R}$ and $H > 0$. If x is any solution of the equation*

$$\left(\frac{x'}{\sqrt{1 + x'^2}} \right)' + a \sin x = |a| + H, \tag{5.2}$$

defined on the interval $[C, D]$ then $D - C < \dfrac{2}{H}$.

Proof. Note that (5.2) is an autonomous equation that admits the energy function

$$E(u, v) := \frac{1}{\sqrt{1 + v^2}} + a \cos(u) + (|a| + H)u, \tag{5.3}$$

which is constant along the solutions of (5.2). Moreover, the potential $V(u) := a \cos(u) + (|a| + H)u$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ since $V'(u) = -a \sin(u) + |a| + H \geq H > 0$ for all $u \in \mathbb{R}$. If x is a solution of (5.2) from a phase plane analysis it follows that x vanishes at only one point, say t_0, where x attains its global minimum. Then, if x is defined for any time $t_1 > t_0$ we have that $x'(t) > 0$ for all $t \in (t_0, t_1]$ and since

$$E(x(t), x'(t)) = E(x(t_0, x'(t_0))) := E_0 \quad \text{for all } t \in [t_0, t_1],$$

by (5.3) we obtain

$$x'(t) = \frac{\sqrt{1 - (E_0 - V(x(t)))^2}}{E_0 - V(x(t))} \quad \text{for all } t \in [t_0, t_1].$$

Thus

$$t_1 - t_0 = \int_{x(t_0)}^{x(t_1)} \frac{E_0 - V(s)}{\sqrt{1 - (E_0 - V(s))^2}} ds.$$

Let us define $u(s) := E_0 - V(s)$ and observe that $u(s) \in (0, 1]$ for all $s \in [x(t_0), x(t_1)]$ and moreover $u'(s) = -V'(s) \leq -H < 0$. Then,

$$t_1 - t_0 \;=\; \int_{x(t_0)}^{x(t_1)} \frac{u(s)}{\sqrt{1-(u(s))^2}} \frac{u'(s)}{u'(s)} ds \;\le\; -\frac{1}{H} \int_{x(t_0)}^{x(t_1)} \frac{u(s)}{\sqrt{1-(u(s))^2}} u'(s) ds$$

$$=\; -\frac{1}{H} \int_{u(x(t_0))}^{u(x(t_1))} \frac{u}{\sqrt{1-u^2}} du \;=\; \frac{1}{H} \int_{u(x(t_1))}^{u(x(t_0))} \frac{u}{\sqrt{1-u^2}} du$$

$$<\; \frac{1}{H} \int_0^1 \frac{u}{\sqrt{1-u^2}} du \;=\; -\frac{1}{H} \sqrt{1-u^2} \, \Big|_{u=0}^{u=1} \;=\; \frac{1}{H}.$$

By an analogous reasoning in case $t_1 < t_0$ we get the desired result. $\qquad\square$

Remark 13. Lemma 12 means that the maximal interval of definition for any solution of (5.2) is finite. Furthermore, the length of any of those maximal intervals is uniformly bounded by the explicit constant $\frac{2}{H}$. To the contrary, note that for $-2|a| \le H \le 0$ the equation (5.2) admit constant solutions, thus defined on the whole real line.

On the other hand, from [6, Corollary 2] it follows that the Dirichlet problem

$$\left(\frac{x'}{\sqrt{1+x'^2}} \right)' = H > 0, \quad u(0) = u(1) = 0,$$

has a solution if and only if $0 < H < 2$. So, the necessary condition given in Lemma 12 for the existence of a solution of (5.2) on an interval $[C, D]$ is sharp.

Theorem 14. *For any $T > 0$ and $a \in \mathbb{R}$, let us consider $H > \frac{4}{T}$ and $e \in \tilde{C}_T$ such that*

$$e(t) = |a| + H \text{ for all } t \in [0, T/2].$$

Then equation (5.1) *has not any solution defined on* $[0, T]$.

Proof. Clearly, (5.1) can not have a solution defined on $[0, T]$ because in that case (5.2) would have a solution defined on $[0, T/2]$, in contradiction with Lemma 12. $\qquad\square$

Corollary 15. *Given $T > 0$, $a \in \mathbb{R}$ and $e \in \tilde{C}_T$ as in Theorem 14 the equation* (5.1) *has not T- periodic solutions.*

Corollary 15 was obtained by other methods in [18], see also [17], but we stress that Theorem 14 can be applied not only for periodic, but also for any other kind of boundary conditions, like for instance Dirichlet or Neumann.

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