NONNEGATIVE OSCILLATIONS FOR A CLASS OF DIFFERENTIAL EQUATIONS WITHOUT UNIQUENESS: A VARIATIONAL APPROACH

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Dedicated to Professor Juan José Nieto on the occasion of his 60th birthday.

Abstract. We deal with the existence of nonnegative and nontrivial solutions T–periodic solutions for the equation $x'' = r(t)x^\alpha - s(t)x^\beta$ where r and s are continous T–periodic functions and $0 < \alpha < \beta < 1$. This equation has been studied in connection with the valveless pumping phenomenon and we will take advantage of its variational structure in order to guarantee its solvability by means of the mountain pass theorem of Ambrosetti and Rabinowitz.

Key words: periodic solution; regular equation; critical point; mountain pass theorem.
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1. Introduction

The second order periodic boundary value problem

$$\text{(1)} \qquad \begin{cases} x''(t) + ax'(t) = r(t)x^\alpha(t) - s(t)x^\beta(t), & t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

where $T > 0$, $a \geq 0$, $r, s \in C(\mathbb{R}/T, \mathbb{R})$ and $0 < \alpha < \beta < 1$, has been firstly studied in [6] in connection with a singular BVP arising as a model for a valveless pumping effect in a simple pipe-tank configuration (see [10] and also [13, Chapter 8] for a nice review of this remarkable phenomenon). Later some other existence results for problem (1) were obtained by means of topological methods, see [4, 5, 13].

The friction term ax' appears in (1) reflecting Poiseuille's law in the original pipe-tank model. On the other hand, the case $a = 0$ is an idealization but not meaningless from the physical (or the mathematical) point of view, as has already been pointed out in [6, Remark 1.4]. Recently, this frictionless case

$$\text{(2)} \qquad \begin{cases} x''(t) = r(t)x^\alpha(t) - s(t)x^\beta(t), & t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

where

(H) $T > 0$, $r, s \in C(\mathbb{R}/T, \mathbb{R})$ and $0 < \alpha < \beta < 1$,

has been studied in [14] by means of the lower/upper solution technique and the averaging method, obtaining in this way sufficient conditions for the existence of positive T-periodic solutions and also for its stability in combination with the Moser twist theorem.

Our aim in this note is to exploit the variational structure of (2) to obtain the existence of a nontrivial nonnegative solution which it is a complementary result to those in [14]. In particular, for positive r and s our main result implies the "universal" solvability of (2) for every $0 < \alpha < \beta < 1$ (compare for instance with [14, Theorem 3.2]). Clearly, $x \equiv 0$ is a nonnegative solution of (2) and our main tool to show that it also exists a nontrivial solution will be the celebrated Mountain Pass Theorem by Ambrosetti and Rabinowitz which we recall for the convenience of the reader.

Theorem 1.1. [9, Theorem 6.4.24] *Let X be a Banach space and let $J \in C^1(X, \mathbb{R})$, $e \in X$ and $R > 0$ such that $\|e\| > R$ and*

$$\inf_{x \in X, \|x\|=R} J(x) > J(0) \geq J(e).$$

Suppose, moreover, that J satisfies the Palais-Smale condition at the level

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

where

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \, \gamma(1) = e\},$$

that is:

$(PS)_c$ *Any sequence $\{x_n\}_{n=1}^{\infty}$ such that $J(x_n) \to c$ and $J'(x_n) \to 0$ has a convergent subsequence.*

Then c is a critical value of J, that is, there exists $x \in X$ such that $J(x) = c$ and $J'(x) = 0$.

The paper is organized as follows: in Section 2 we analyze the autonomous version of (2). This will allow us to obtain a picture of what is going on that will be useful to deal in Section 3 with the general case by means of a variational method.

In the sequel, for a continuous function h on $[0, T]$ we shall denote its minimum by h_*, its maximum by h^* and its mean value by $\bar{h} = \dfrac{1}{T}\displaystyle\int_0^T h(s)ds$.

2. The autonomous case

In order to gain insight into problem (2) we shall consider in this section the problem with constant functions r and s, that is, we deal with

$$\left\{ \begin{array}{l} x''(t) = r x^{\alpha}(t) - s x^{\beta}(t), \quad t \in [0, T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{array} \right. \tag{3}$$

where r, $s > 0$ and $0 < \alpha < \beta < 1$. Since the equation in (3) is conservative it admits the energy integral

$$\frac{x'^2}{2} + V(x) = c, \tag{4}$$

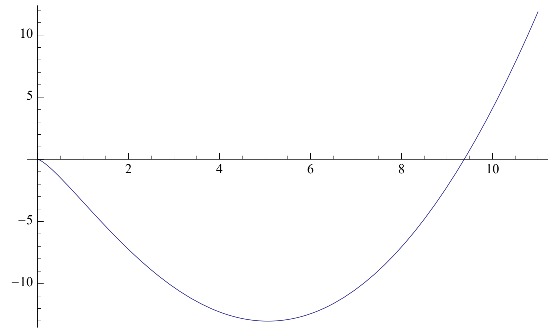

FIGURE 1. Graph of the potential V for the values $r = 12$, $s = 8$, $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{4}$.

where the potential is given by

$$V(x) = -r\frac{x^{\alpha+1}}{\alpha+1} + s\frac{x^{\beta+1}}{\beta+1}, \quad x \geq 0.$$

The potential V attains its absolute minimum at $\bar{x} = \left(\frac{r}{s}\right)^{\frac{1}{\beta-\alpha}}$ and for $c \in (V(\bar{x}), 0]$ the equation $V(x) = c$ has exactly two solutions. So, V exhibits a "potential well" and provides the following information about the orbits in the (x, x') phase-plane: the only positive equilibrium, namely $\bar{x}$, is a center surrounded by closed orbits. For the energy level $c \in (V(\bar{x}), 0)$ the orbit of the corresponding periodic solution $x_c(t)$ of (3) crosses the x-axis at points $(A_c, 0)$ and $(B_c, 0)$ and $A_c \leq x_c(t) \leq B_c$ for all $t \in \mathbb{R}$, where $0 < A_c < \bar{x} < B_c$ are the solutions of $V(A_c) = V(B_c) = c$. The minimal period $p(c)$ of x_c is given by the formula

$$p(c) = \sqrt{2}\int_{A_c}^{B_c} \frac{1}{\sqrt{c - V(x)}}dx. \tag{5}$$

By linearization at $\bar{x}$, it is easy to see that

$$\lim_{c\to V(\bar{x})} p(c) = \frac{2\pi}{\sqrt{V''(\bar{x})}} = 2\pi\sqrt{\frac{r^{\frac{1-\beta}{\beta-\alpha}}}{(\beta-\alpha)s^{\frac{1-\alpha}{\beta-\alpha}}}}. \tag{6}$$

On the other hand, letting B_0 denote the positive solution of the equation $V(x) = 0$, it turns out that the orbit through the initial condition $(B_0, 0)$ reaches the origin in finite time, therefore defining another periodic solution with period

$$p(0) = \sqrt{2}\int_0^{\left(\frac{r(\beta+1)}{s(\alpha+1)}\right)^{\frac{1}{\beta-\alpha}}} \frac{dx}{\sqrt{r\frac{x^{\alpha+1}}{\alpha+1} - s\frac{x^{\beta+1}}{\beta+1}}} \tag{7}$$

Since V' is not regular at $x = 0$ it is not clear whether p is continuous at $c = 0$. We prove this is indeed true in the following result.

Lemma 2.1. *The period function p is continuous in $[V(\bar{x}), 0]$.*

Proof. Let us fix $0 < \tilde{x} < \bar{x}$ and split

$$\int_{A_c}^{B_c} \frac{1}{\sqrt{c - V(x)}} dx = \int_{A_c}^{\tilde{x}} \frac{1}{\sqrt{c - V(x)}} dx + \int_{\tilde{x}}^{B_c} \frac{1}{\sqrt{c - V(x)}} dx.$$

Having into account known results ([11], Prop. 3.1.2), it suffices to prove that

$$\phi(c) = \int_{A_c}^{\tilde{x}} \frac{1}{\sqrt{c - V(x)}} dx$$

is continuous at $c = 0$. Letting $\epsilon = |c|$ and $W = |V|$, we have to show that

$$\lim_{\epsilon \to 0^+} \epsilon \int_{A_{-\epsilon}}^{\tilde{x}} \frac{dx}{W(x)\sqrt{W(x) - \epsilon} + (W(x) - \epsilon)\sqrt{W(x)}} = 0 \tag{8}$$

With the change of variable $W(x) = \epsilon t$, we obtain

$$\int_{A_{-\epsilon}}^{\tilde{x}} \frac{dx}{W(x)\sqrt{W(x) - \epsilon} + (W(x) - \epsilon)\sqrt{W(x)}} = \frac{1}{\sqrt{\epsilon}} \int_{1}^{W(\tilde{x})/\epsilon} \frac{dt}{(t\sqrt{t-1} + (t-1)\sqrt{t})W'(W^{-1}(\epsilon t))} \tag{9}$$

Let $k = \alpha + 1$. Since the function $z \mapsto \dfrac{W'(W^{-1}(z))}{z^{1-1/k}}$ is bounded away from 0 for $z \in]0, W(\tilde{x})]$, we may write

$$W'(W^{-1}(\epsilon t)) = \epsilon^{1-1/k}\theta(\epsilon, t) \tag{10}$$

where θ is bounded away from 0 for $t \in]1, W(\tilde{x})/\epsilon]$ uniformly with respect to $\epsilon > 0$. From (8)-(9)-(10) and the fact that the integral $\displaystyle\int_1^{+\infty} \frac{dt}{t\sqrt{t-1} + (t-1)\sqrt{t}}$ converges, it follows that

$$\epsilon \int_{A_{-\epsilon}}^{\tilde{x}} \frac{dx}{W(x)\sqrt{W(x) - \epsilon} + (W(x) - \epsilon)\sqrt{W(x)}} = O(\epsilon^{\frac{1}{k} - \frac{1}{2}})$$

as $\epsilon \to 0^+$. Since $k < 2$ we are done.

□

So the analysis of the simple autonomous problem (3) lead us to the following interesting observations:

- *Multiplicity*: for every period $T > 0$, problem (3) always has the constant nontrivial solution $\bar{x}$ and moreover for the periods $T_c := p(c)$, $c \in [V(\bar{x}), 0]$ that constant solution coexists with another non constant T_c-periodic solution, namely x_c. To our knowledge this is the first appearance of a multiplicity result related to (2) .
- *Homoclinics*: the orbit of (3) trough $(B_0, 0)$ may be seen as a homoclinic solution connecting $(0, 0)$ to itself in a finite time $\tilde{T} = p(0)$ (see e.g. [7, 8, 15] for other settings where this phenomenon appears). Therefore the homoclinic can be continued as 0 in an interval of arbitrary length and then repeated periodically (note that this situation is possible only because $(0, 0)$ is a point of non-uniqueness for (3)). In this way, for each $T \geq \tilde{T}$ there exists a non-constant and nonnegative solution $x(t)$ of (3) that vanishes at some points.

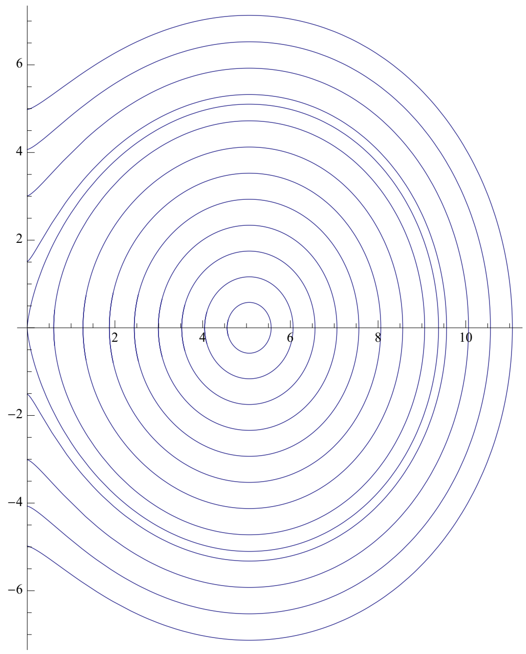

FIGURE 2. Phase plane for (3) with the values $r = 12$, $s = 8$, $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{4}$.

Remark 2.1. *The study of the properties of the period function p is far from trivial and it has been the subject of a great ammount of research in last decades, see [1, 2, 3, 12] and the references therein.*

Using the same notation as in [3, Corollary 2.6] we have that in problem (3)

$$\nabla = \left(\frac{r}{s}\right)^{\frac{4-2\beta}{\alpha-\beta}} s^2(\alpha-\beta)^2\left(2\alpha^2+\alpha(7\beta-1)+(\beta-1)(2\beta+1)\right).$$

Now, if we take $\beta = \frac{1+\alpha}{2}$ we obtain

$$sgn(\nabla) = sgn(6\alpha^2+3\alpha-1),$$

and then we are able to conclude (see also [1, Example 2]) that:

- *For $0 < \alpha < 0.228714$ and $\beta = \frac{1+\alpha}{2}$ the function p is decreasing on some interval $(V(\bar{x}), V(\bar{x})+\delta_1)$ with $\delta_1 > 0$.*
- *For $0.228714 < \alpha < 1$ and $\beta = \frac{1+\alpha}{2}$ the function p is increasing on some interval $(V(\bar{x}), V(\bar{x})+\delta_2)$ with $\delta_2 > 0$.*

3. THE NON-AUTONOMOUS CASE

Let us introduce the following modified problem

$$\begin{cases} x''(t) = r(t)|x(t)|^{\alpha-1}x(t) - s(t)x_+(t)^\beta, & t\in[0,T], \\ x(0)=x(T), \quad x'(0)=x'(T). \end{cases} \tag{11}$$

Notice that a nonnegative solution of (11) is also a solution of (2) and that the solutions of (11) correspond to the critical points of the functional $J : H_T^1 \to \mathbb{R}$ given by

$$J(x) = \int_0^T \left(\frac{x'(t)^2}{2} + r(t)\frac{|x(t)|^{\alpha+1}}{\alpha+1} - s(t)\frac{x_+(t)^{\beta+1}}{\beta+1} \right) dt, \tag{12}$$

where

$$H_T^1 = \{x \in AC[0,T] : x' \in L^2[0,T],\, x(0) = x(T)\}$$

is a Hilbert space with the inner product

$$(x,y)_{H_T^1} = \int_0^T x(t)y(t)dt + \int_0^T x'(t)y'(t)dt.$$

Clearly, $J \in C^1(H_T^1, \mathbb{R})$ and for each $x, v \in H_T^1$ we have

$$J'(x)v = \int_0^T \left(x'(t)v'(t) + r(t)|x(t)|^{\alpha-1}x(t)v(t) - s(t)x_+(t)^{\beta}v(t) \right) dt. \tag{13}$$

Our next aim is to show the existence of another critical point of J, different from the trivial one, by means of Theorem 1.1.

Theorem 3.1. *If $0 < \alpha < \beta < 1$, $r_* > 0$ and $s_* > 0$ then the functional J given by (12) has a nontrivial critical point in H_T^1.*

Proof. A word about notation is in order: in the sequel K shall denote a positive constant ($K > 0$) that can change from time to time and whose exact value is not relevant for our computations.

Taking into account that

$$r(t)\frac{|x(t)|^{\alpha+1}}{\alpha+1} - s(t)x_+(t)^{\beta+1} \geq r_*\frac{|x(t)|^{\alpha+1}}{\alpha+1} - s^* x_+(t)^{\beta+1} = r_*\frac{|x(t)|^{\alpha+1}}{\alpha+1}\left(1 - s^*\frac{\alpha+1}{r_*}x_+(t)^{\beta-\alpha}\right),$$

and $0 < \alpha < \beta < 1$ we have that for $\|x\|_\infty$ small enough

$$r(t)\frac{|x(t)|^{\alpha+1}}{\alpha+1} - s(t)x_+(t)^{\beta+1} \geq K\,|x(t)|^{\alpha+1} \geq K\,|x(t)|^2.$$

Then, from the continuous embedding of H_T^1 into $C[0,T]$ the following property holds:

(14) There exists $R_1 > 0$ such that if $\|x\|_{H_T^1} \leq R_1$ then $J(x) \geq K\,\|x\|_{H_T^1}^2$

Since $J(0) = 0$ it is clear from (14) that $x \equiv 0$ is a strict local minimum of J. On the other hand, if $c \in \mathbb{R}$, $c > 0$, then

$$Jc = \frac{T\bar{r}}{\alpha+1}c^{\alpha+1} - T\bar{s}c^{\beta+1},$$

so taking into account that $\alpha < \beta$ and $\bar{s} > 0$ we have

$$\lim_{c \to +\infty} Jc = -\infty. \tag{15}$$

From (14) and (15) it is clear that J exhibits a "mountain pass" geometry, that is, we have proved:

Claim 1.- There exist $R > 0$ and $\delta > 0$ such that if $\|x\|_{H_T^1} = R$ then $J(x) \geq \delta$.

Claim 2.- There exists $e \in \mathbb{R}$, $e > R$, such that $J(e) < 0$.

So, in order to get a nontrivial critical point of J it remains to verify the Palais-Smale condition, providing in this way the necessary compactness.

Claim 3.- (PS) If $\{x_n\}_{n=1}^{\infty} \subset H_T^1$ is such that $\{J(x_n)\}_{n=1}^{\infty}$ is bounded and $J'(x_n) \to 0$ then $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Let us write $x_n = w_n + h_n$ where $\int_0^T w_n(t)dt = 0$ and $h_n \in \mathbb{R}$. By (13) we have

$$J'(x_n)w_n = \int_0^T \left(w_n'(t)^2 + r(t)|x_n(t)|^{\alpha-1}x_n(t)w_n(t) - s(t)x_{n+}(t)^{\beta}w_n(t)\right)dt, \tag{16}$$

and by assumption

$$|J'(x_n)w_n| \le \varepsilon_n \|w_n\|_{H_T^1} \text{ with } \varepsilon_n \to 0. \tag{17}$$

On the other hand,

$$|r(t)|x_n(t)|^{\alpha-1}x_n(t)w_n(t)| \le r^*|x_n(t)|^{\alpha}|w_n(t)| \le r^*|w_n(t) + h_n|^{\alpha}|w_n(t)|$$
$$\le K\left(|w_n(t)|^{\alpha+1} + |h_n|^{\alpha}|w_n(t)|\right) \le \varepsilon|w_n(t)|^2 + K\left(|w_n(t)|^{\alpha+1} + |h_n|^{2\alpha}\right),$$

where we have used the inequality $|x+y|^{\alpha} \le K(|x|^{\alpha} + |y|^{\alpha})$ for $x, y \in \mathbb{R}$ (the constant depending only on α) and Young's inequality with ε. Now, by using Holder's inequality

$$\left|\int_0^T r(t)|x_n(t)|^{\alpha-1}x_n(t)w_n(t)dt\right| \le \varepsilon\|w_n\|_2^2 + K(\|w_n\|_2^{\alpha+1} + |h_n|^{2\alpha}),$$

so

$$-\varepsilon\|w_n\|_2^2 - K(\|w_n\|_2^{\alpha+1} + |h_n|^{2\alpha}) \le \int_0^T r(t)|x_n(t)|^{\alpha-1}x_n(t)w_n(t)dt. \tag{18}$$

In a similar way

$$-\varepsilon\|w_n\|_2^2 - K(\|w_n\|_2^{\beta+1} + |h_n|^{2\beta}) \le -\int_0^T s(t)x_{n+}(t)^{\beta}w_n(t)dt. \tag{19}$$

From (16), (17), (18) and (19) it follows that

$$\|w_n'\|_2^2 - 2\varepsilon\|w_n\|_2^2 - K(\|w_n\|_2^{\alpha+1} + \|w_n\|_2^{\beta+1} + |h_n|^{2\alpha} + |h_n|^{2\beta}) \le \varepsilon_n\|w_n\|_{H_T^1}, \tag{20}$$

and then

$$\|w_n\|_{\infty}^2 \le K(\|w_n\|_2^{\alpha+1} + \|w_n\|_2^{\beta+1} + |h_n|^{2\alpha} + |h_n|^{2\beta}). \tag{21}$$

If $h_n \to +\infty$ then from (20) we deduce that $x_n \to +\infty$ uniformly and in that case we obtain the following contradiction

$$J'(x_n)1 = \int_0^T \left(r(t)|x_n(t)|^{\alpha-1}x_n(t) - s(t)x_{n+}(t)^{\beta}\right)dt \to -\infty.$$

If $h_n \to -\infty$ we get a contradiction in an analogous way, so we can suppose that $|h_n|$ is bounded. Then by (20) and (21) it follows that $\{w_n\}$ is bounded in H_T^1 which implies that $\{x_n\}$ is also bounded in H_T^1. Then, up to a subsequence, $\{x_n\}$ converges weakly in H_T^1 (and uniformly on $[0, T]$) to some $x \in H_T^1$. Consequently

$$\lim_{n\to\infty} J'(x)(x - x_n) - J'(x_n)(x - x_n) = 0.$$

Setting $F(t,x) = r(t)|x|^{\alpha-1}x - s(t)x_+^\beta$ and taking into account that

$$J'(x)(x-x_n)-J'(x_n)(x-x_n) = \int_0^T (x'(t)-x_n'(t))^2+(F(t,x(t))-F(t,x_n(t)))(x(t)-x_n(t))dt,$$

and

$$\lim_{n\to\infty}\int_0^T (F(t,x(t)-F(t,x_n(t)))(x(t)-x_n(t))dt = 0,$$

we have

$$\lim_{n\to\infty}\int_0^T (x'(t)-x_n'(t))^2 dt = 0,$$

and therefore $x_n \to x$ in H_T^1. □

As consequence of the previous theorem we obtain our main result which is complementary to [14, Theorem 3.2 and Corollary 1].

Theorem 3.2. *If $0 < \alpha < \beta < 1$, $r_* > 0$ and $s_* > 0$ then problem (2) has a nontrivial solution such that $x(t) \geq 0$ for all $t \in [0,T]$.*

Proof. From Theorem 3.1 it follows the existence of a nontrivial critical point x of J which, equivalently, is a solution of the modified problem (11). If we suppose that

$$x(t_0) = \min_{t\in[0,T]} x(t) < 0,$$

then we get $x''(t_0) = r(t_0)|x(t_0)|^{\alpha-1}x(t_0) < 0$, a contradiction. Hence, $x(t) \geq 0$ for all $t \in [0,T]$ and then it is also a nontrivial nonnegative solution of problem (2). □

Remark 3.1. *The condition $r_* > 0$ in Theorem 3.2 is not necessary in order to get a positive solution for problem (2) as the following example, taken from [5], shows: let us define*

$$r(t) = 6.6 - 5.7\cos(t) - 9\cos^2(t) \qquad \textit{and} \qquad s(t) = 0.3.$$

Then $r_* = r(0) = -8.1 < 0$ but it is easy to check that $x(t) = (2+\cos(t))^3 > 0$ is a solution of the problem

$$\begin{cases} x''(t) = r(t)x^{1/3}(t) - s(t)x^{2/3}(t), & t \in [0,2\pi], \\ x(0) = x(2\pi), \quad x'(0) = x'(2\pi). \end{cases} \tag{22}$$

So it would be interesting to remove the assumption $r_* > 0$, but our approach seems not work whenever $r_* \leq 0$.

In the following result we provide some "a priori" estimates for the non-trivial solutions of problem (2), that could be useful to localize the solutions given by Theorem 3.2.

Lemma 3.3. *Let $x \not\equiv 0$ be a solution of problem (2). If $\alpha < \beta$, $r_* > 0$ and $s_* > 0$ then the following claims hold:*

(i) $\|x\|_\infty \geq \left(\dfrac{r_*}{s^*}\right)^{\frac{1}{\beta-\alpha}}$.

(ii) $\|x'\|_\infty \leq s^*\, s_*^{-\frac{\beta}{\beta-\alpha}} \displaystyle\int_0^T r(t)^{\frac{\beta}{\beta-\alpha}}$.

Proof. Let $t_0 \in [0,T]$ such that $x(t_0) = \|x\|_\infty > 0$. Then

$$x''(t_0) = r(t_0)x^\alpha(t_0) - s(t_0)x^\beta(t_0) \le 0.$$

Hence,

$$r_* x^\alpha(t_0) \le s^* x^\beta(t_0),$$

and claim (i) follows.

To prove (ii) take into account that for all $t \in [0,T]$

$$x'(t) = x'(t_0) + \int_{t_0}^t x''(\tau)d\tau = \int_{t_0}^t r(\tau)x^\alpha(\tau) - s(\tau)x^\beta(\tau)d\tau \le \int_{t_0}^t r(\tau)x^\alpha(\tau)d\tau,$$

and hence

$$\|x'\|_\infty \le \int_0^T r(\tau)x^\alpha(\tau)d\tau. \tag{23}$$

On the other hand, integrating the equation of problem (2) and taking into account the boundary conditions we have

$$\int_0^T r(\tau)x^\alpha(\tau)d\tau = \int_0^T s(\tau)x^\beta(\tau)d\tau. \tag{24}$$

From (24) and Hölder's inequality it follows

$$s_* \int_0^T x^\beta(\tau)d\tau \le \int_0^T r(\tau)x^\alpha(\tau)d\tau \le \|r\|_{\frac{\beta}{\beta-\alpha}} \left(\int_0^T x^\beta(\tau)d\tau\right)^{\frac{\alpha}{\beta}},$$

and therefore

$$\int_0^T x^\beta(\tau)d\tau \le \left(\frac{\|r\|_{\frac{\beta}{\beta-\alpha}}}{s_*}\right)^{\frac{\beta}{\beta-\alpha}}. \tag{25}$$

Now, by using (24) and (25) we arrive at

$$\int_0^T r(\tau)x^\alpha(\tau)d\tau \le s^* \int_0^T x^\beta(\tau)d\tau \le s^* \left(\frac{\|r\|_{\frac{\beta}{\beta-\alpha}}}{s_*}\right)^{\frac{\beta}{\beta-\alpha}},$$

which together with (23) implies (ii).

□

Remark 3.2. *The a priori estimate (i) in Lemma 3.3 is sharp as the phase-plane analysis of the autonomous problem (3), given in Section 2, shows.*

Furthermore, for problem (3) the conservation of the energy implies easily the following relation between the maximum norms of x and x',

$$\|x'\|_\infty = \sqrt{2(V(\|x\|_\infty) - V(\bar{x}))}.$$

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References

[1] C. Chicone, The monotonicity of the period function for planar Hamiltonian vector fields, *J. Differential Equations* **69** (1987) 310-321.

[2] A. R. Chouikha, Monotonicity of the period function for some planar differential systems. I. Conservative and quadratic systems, *Appl. Math.* **32** (2005), 305-325.

[3] S. N. Chow and D. Wang, On the monotonicity of the period function of some second order equations, *Časopis Pěst. Mat.* **111** (1986) 14-25.

[4] J. A. Cid, G. Infante, M. Tvrdý and M. Zima, A topological approach to periodic oscillations related to the Liebau phenomenon, *J. Math. Anal. Appl.* **423** (2015) 1546–1556.

[5] J. A. Cid, G. Infante, M. Tvrdý and M. Zima, New results for the Liebau phenomenon via fixed point index, *Nonlinear Anal. Real World Appl.* **35** (2017), 457–469.

[6] J. A. Cid, G. Propst and M. Tvrdý, On the pumping effect in a pipe/tank flow configuration with friction, *Phys. D* **273-274** (2014) 28-33.

[7] I. Coelho and L. Sanchez, Travelling wave profiles in some models with nonlinear diffusion, *Appl. Math. Comput.* **235** (2014), 469-481.

[8] A. Gavioli and L. Sanchez, Positive homoclinic solutions to some Schrödinger type equations, *Differential Integral Equations* **29** (2016), 665-682.

[9] P. Drábek and J. Milota, *Methods of nonlinear analysis. Applications to differential equations*, Birkhäuser Advanced Texts, 2007.

[10] G. Propst, Pumping effects in models of periodically forced flow configurations, *Phys. D* **217** (2006), 193–201.

[11] R. Schaaf, *Global Solution Branches of Two Point Boundary Value Problems*, Lecture Notes in Mathematics, Vol. 1458, Springer, Berlin, 1990; Habilitationsschrift, Ruprecht-Karls-Universitt, Heidelberg (1987).

[12] A. Sfecci, From isochronous potentials to isochronous systems, *J. Differential Equations* **258** (2015) 1791-1800.

[13] P. J. Torres, *Mathematical models with singularities*, Atlantis Briefs in Differential Equations, vol. 1, 2015.

[14] F. Wang, J. A. Cid and M. Zima, Lyapunov stability for regular equations and applications to the Liebau phenomenon, *Discrete Contin. Dyn. Syst.* **38** (2018) 4657–4674.

[15] Zhou, Wenshu; Qin, Xulong; Xu, Guokai; Wei, Xiaodan, On the one-dimensional p-Laplacian with a singular nonlinearity, *Nonlinear Anal.* **75** (2012), 3994–4005.