Existence of infinitely many solutions for second–order singular initial value problems with an application to nonlinear massive gravity *

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Abstract

We prove the existence of infinitely many solutions for a second–order singular initial value problem between given lower and upper solutions. Our study is motivated by a singular problem which arises in the field of nonlinear massive gravity. Moreover, we also discuss the global behavior of solutions of the motivating problem. Our arguments lean at some steps on lower and upper solutions with corners in their graphs, thus showing the applicability of this more general definition of lower and upper solutions in the analysis of a concrete mathematical model.

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1 Introduction

Physical models in which the laws of General Relativity change at cosmological distances (*infrared–modified gravity*) have been recently proposed in order to explain the observed accelerated expansion of the Universe. One simple possibility to get an infrared–modified gravity is to assume a non–zero mass for the graviton, bringing about the so–called *massive gravity theories* described, for instance, in the survey [23].

Babichev, Deffayet and Ziour study in [7, section 4] static spherically symmetric solutions in the associated decoupling limit of nonlinear Pauli–Fierz theories, the simplest massive gravity models. In that context the mathematical questions of existence and multiplicity of positive solutions for the following problem arise:

$$y'' = \frac{\sqrt[3]{y}}{4\sqrt[3]{x^8}} - \frac{1}{6\sqrt[3]{x^5 y}} \quad (x > 0), \quad y(0) = y'(0) = 0. \tag{1.1}$$

Problem (1.1) corresponds to equations (4.25) and (4.26) in [7] with the AGS potential ($s = +1$), see the discussion in [7, page 33].

In this paper we present a general result on existence of infinitely many positive solutions for second–order singular initial value problems which, in particular, guarantees the existence of infinitely many positive solutions to Problem (1.1). Moreover, we show that some of these solutions are defined on $[0, +\infty)$ and some of them are not, and we give some additional information about the set of solutions.

It is remarkable that, in agreement with our results, Jean Écalle, employing the resurgence theory, provided[1] the authors of [7] with an unpublished proof of the existence of infinitely many solutions to the Problem (1.1).

[1]Personal communication of Cedric Deffayet.

From all the characteristics of this paper, we single out the following four, which we judge of interest:

- First, our general theorem (Theorem 2.2) is based on a well–known classic result on ordinary differential equations, thus it is easily understandable for a broad part of the mathematical scientific community.

- Second, Theorem 2.2 seems to fill a gap in the existing literature on singular initial value problems, since, as far as the authors are aware, it is the first one which ensures *infinitely* many solutions between a pair of lower and upper solutions, and it is also the first one which allows non–integrable singularities around $x = 0$.

- Third, we give some supplementary information about the solutions of Problem (1.1), studying in particular their continuability up to $+\infty$ and their localization near $x = 0$.

- Fourth, lower and upper solutions with corners in their graphs naturally arise in our study of the solutions of Problem (1.1) in Section 3. We can quote many references in connection with non–differentiable lower and upper solutions for second–order differential equations, see [1, 9, 12, 13, 15, 16, 17, 21, 22], but examples of their applicability in the study of mathematical models were demanding.

Note that the right–hand side of the differential equation in (1.1) exhibits singularities not only in $y = 0$ but also in $x = 0$, and therefore it immediately falls outside the scope of most existence results, such as [3, Theorem 2.1], [4, Section 2.3], [5, Theorem 2.2], [25, Theorem 3.4] (furthermore, only positive right–hand sides are considered in these references, which is not the case in (1.1) either) or [6]. On the other hand, there seems to be very few existence results which allow singularities both in $x = 0$ and $y = 0$, and the ones we know do not apply because they rely to some extent on integrability with respect to x around 0, so they do not apply for Problem (1.1). Note, for instance, that the right–hand side of the differential equation in (1.1) does not satisfy condition

(2.14) in [2, Corollary 2], as the corresponding integral diverges near 0 for any $c > 0$, and it does not satisfy the condition (f0) on page 521 of [10] either.

The main result in this article on existence of positive solutions for singular problems uses lower and upper solutions. Remarkably, we construct successive approximations to singular Cauchy problems by solving non–singular Dirichlet problems.

We organize this paper as follows: first (Section 2), we introduce our definitions of lower and upper solutions and we state and prove our main results for second–order initial value problems in general; then (Section 3) we restrict our attention to the particular case of Problem (1.1), and we use the results in Section 2 to deduce the existence of infinitely many solutions for (1.1), to give some information about their domains and their localization between explicitly given lower and upper solutions, and to study the set of solutions.

2 Theoretical results

Let $L > 0$ be fixed and consider the nonlocal initial value problem

$$y'' = f(x, y) \text{ for all } x \in I = (0, L], \quad y(0) = 0, \;\; y'(0) = y_1 \geq 0, \tag{2.2}$$

where $f : I \times (0, +\infty) \to \mathbb{R}$ is continuous. We look for solutions which are continuous on $\overline{I} = [0, T]$, differentiable on $[0, L)$, and twice continuously differentiable on $(0, L)$.

We follow De Coster and Habets [8] and we introduce the following definition.

DEFINITION 2.1 *A lower solution of (2.2) is a function $\alpha \in \mathcal{C}(\bar{I})$, $\alpha > 0$ on I, such that*

(a1) for any $x_0 \in (0, L)$, either $\alpha'_-(x_0) < \alpha'_+(x_0)$, or there exists an open interval $I_0 \subset (0, L)$ such that $x_0 \in I_0$, $\alpha \in \mathcal{C}^2(I_0)$ and, for all $x \in I_0$,

$$\alpha''(x) \geq f(x, \alpha(x));$$

(a2) $\alpha(0) = 0$ and there exists $\alpha'(0) = y_1$.

An upper solution of (2.2) is a function $\beta \in \mathcal{C}(\bar{I})$, $\beta > 0$ on I, such that

(b1) for any $x_0 \in (0, L)$, either $\beta'_-(x_0) > \beta'_+(x_0)$, or there exists an open interval $I_0 \subset (0, L)$ such that $x_0 \in I_0$, $\beta \in \mathcal{C}^2(I_0)$ and, for all $x \in I_0$,

$$\beta''(x) \le f(x, \beta(x));$$

(b2) $\beta(0) = 0$ and there exists $\beta'(0) = y_1$.

Finally, a solution of (2.2) is a function which is both a lower and an upper solution.

Remark 2.1 *An important particular case of Definition 2.1 is that of lower and upper solutions in the classical sense, which belong to $\mathcal{C}^2(0, L)$ and satisfy the corresponding differential inequalities on the whole of $(0, L)$.*

Notice that Definition 2.1 allows lower (upper) solutions to have a number of downwards (upwards) corners in their graphs. Classical lower and upper solutions are not satisfactory in this paper, as certain lower and upper solutions with corners are involved in our arguments in Section 3.

According to [8], non–differentiable lower and upper solutions for second–order equations were considered first by Picard [20], and later rediscovered by Nagumo [19]. We can quote many more recent references featuring lower and upper solutions with corners in their graphs, but probably the definitions introduced by De Coster and Habets in [8] best balance generality and simplicity. Definition 2.1 is even simpler, and it suffices for the purposes in this paper.

We will need the following result, which goes back to Scorza Dragoni [24] and Nagumo [18] when classical lower and upper solutions are considered, about the existence of solution for a regular Dirichlet boundary value problem between well–ordered lower and upper solutions, see [8, Theorem 1.3].

Theorem 2.1 *Let $\alpha, \beta \in \mathcal{C}([a, b])$ be such that $\alpha \le \beta$ on $[a, b]$ and consider the compact set $E = \{(x, y) : a \le x \le b,\ \alpha(x) \le y \le \beta(x)\}$.*

If $f : E \longrightarrow \mathbb{R}$ is continuous and α and β satisfy, respectively, $(a1)$ and $(b1)$ in Definition 2.1 with $(0, L)$ replaced by (a, b), then for each $y_a \in [\alpha(a), \beta(a)]$ and each $y_b \in [\alpha(b), \beta(b)]$ the Dirichlet problem

$$y''(x) = f(x, y(x)) \text{ for all } x \in (a, b), \quad y(a) = y_a, \; y(b) = y_b,$$

has a solution $y \in \mathcal{C}^2([a, b])$ such that $\alpha(x) \leq y(x) \leq \beta(x)$ for all $x \in [a, b]$.

Next we present our main result on existence of solutions to Problem (2.2):

Theorem 2.2 *Let α and β be, respectively, a lower and an upper solution of Problem (2.2).*

If $\alpha(x) \leq \beta(x)$ for all $x \in \overline{I}$, then for each $y_L \in [\alpha(L), \beta(L)]$ the Problem (2.2) has a solution such that $y(L) = y_L$ and $\alpha(x) \leq y(x) \leq \beta(x)$ for all $x \in \overline{I}$.

Proof. Let $y_L \in [\alpha(L), \beta(L)]$ be fixed. For each $n \in \mathbb{N}$, $n \geq 2$, we consider the Dirichlet problem

$$y''(x) = f(x, y) \text{ for all } x \in (L/n, L), \quad y(L/n) = \alpha(L/n), \quad y(L) = y_L, \quad (2.3)$$

for which the restricted functions $\alpha_{|[L/n,L]}$ and $\beta_{|[L/n,L]}$ are a pair of well–ordered lower and upper solutions in the sense of Theorem 2.1.

Note that Problem (2.3) is not singular between $\alpha_{|[L/n,L]}$ and $\beta_{|[L/n,L]}$, so Theorem 2.1 guarantees that (2.3) has a solution $\tilde{y}_n : [L/n, L] \to \mathbb{R}$ between $\alpha_{|[L/n,L]}$ and $\beta_{|[L/n,L]}$. Finally, we define a continuous function $y_n : \bar{I} \to \mathbb{R}$ as follows:

$$y_n = \alpha \text{ on } [0, L/n], \quad \text{and} \quad y_n = \tilde{y}_n \text{ on } [L/n, L].$$

The definition implies that the sequence $\{y_n\}_{n \in \mathbb{N}}$ is uniformly bounded on $\bar{I}$. Next we show that the sequence is uniformly equicontinuous on $\bar{I}$. To do so, let $\varepsilon > 0$ be fixed and let $L_\varepsilon \in (0, L/2)$ be so small that

$$0 < \beta(z) < \varepsilon/4 \quad \text{for all } z \in (0, L_\varepsilon]. \quad (2.4)$$

Take $n_1 \in \mathbb{N}$, $n_1 \geq 2$, such that $L/n < L_\varepsilon$ whenever $n > n_1$ and let $\delta_1 > 0$ be such that for all $x_1, x_2 \in [0, L]$ the relation $|x_1 - x_2| < \delta_1$ implies

$$|y_n(x_1) - y_n(x_2)| < \varepsilon/2 \quad \text{for all } n \in \{2, 3, \ldots, n_1\}. \quad (2.5)$$

Now for $n \in \mathbb{N}$, $n > n_1$, and for all $x \in [L_\varepsilon, L]$ we have

$$|y_n''(x)| = |\tilde{y}_n''(x)| = |f(x, y_n(x))| \leq \max_{(s,z)\in K} |f(s,z)|, \tag{2.6}$$

where $K = \{(s,z) \in [L_\varepsilon, L] \times (0, +\infty) : \alpha(s) \leq z \leq \beta(s)\}$ is compact. By the mean value theorem, there exists $\xi_n \in (L_\varepsilon, L)$ such that

$$|y_n'(\xi_n)| = \left| \frac{y_L - y_n(L_\varepsilon)}{L - L_\varepsilon} \right| \leq \frac{2 \max_{0\leq x\leq L} \beta(x)}{L/2} = \frac{4}{L} \max_{0\leq x\leq L} \beta(x),$$

which, together with (2.6), implies that $\{y_n' : n > n_1\}$ is uniformly bounded on $[L_\varepsilon, L]$. Therefore, there exists $\delta_2 > 0$ such that for all $x_1, x_2 \in [L_\varepsilon, L]$ the condition $|x_1 - x_2| < \delta_2$ implies that

$$|y_n(x_1) - y_n(x_2)| < \varepsilon/2 \quad \text{for all } n \in \mathbb{N},\, n > n_1. \tag{2.7}$$

Finally, we prove that for $\delta = \min\{\delta_1, \delta_2\}$ we have

$$x_1, x_2 \in \bar{I},\ |x_1 - x_2| < \delta \ \Rightarrow\ |y_n(x_1) - y_n(x_2)| < \varepsilon \quad \text{for all } n \in \mathbb{N},\, n \geq 2. \tag{2.8}$$

To prove that (2.8) holds we distinguish three cases:

Case 1: $x_1, x_2 \in [0, L_\varepsilon]$. For all $n \in \mathbb{N}$, $n \geq 2$, the triangle inequality and (2.4) yield

$$|y_n(x_1) - y_n(x_2)| < \beta(x_1) + \beta(x_2) < \varepsilon/2 < \varepsilon.$$

Case 2: $x_1, x_2 \in [L_\varepsilon, L]$. In this case (2.8) is guaranteed by virtue of (2.7) for $n > n_1$, and by (2.5) for $n \in \{2, 3, \ldots, n_1\}$.

Case 3: $x_1 < L_\varepsilon < x_2$. This can be reduced to the previous cases, as for all $n \in \mathbb{N}$, $n \geq 2$, we have

$$|y_n(x_1) - y_n(x_2)| \leq |y_n(x_1) - y_n(L_\varepsilon)| + |y_n(L_\varepsilon) - y_n(x_2)| < \varepsilon.$$

The Ascoli–Arzelá Theorem ensures that $\{y_n\}_{n=2}^\infty$ has a subsequence, that we denote again by $\{y_n\}_{n=2}^\infty$, which converges uniformly to some continuous function y on $\bar{I}$ such that $y(L) = y_L$. Obviously, $\alpha \leq y \leq \beta$ on $\bar{I}$, which implies that $y(0) = 0$ and that there exists $y'(0) = y_1$. Standard arguments reveal that y satisfies the differential equation on $[r, L]$ for all $r \in (0, L)$, so y solves (2.2).

□

Remark 2.2 *Theorem 2.2 guarantees the existence of infinitely many solutions to Problem (2.2) when* $\alpha(\tilde{L}) < \beta(\tilde{L})$ *for some* $\tilde{L} \in (0, L]$.

We also notice that the differential inequalities used in our definitions of lower and upper solutions for Problem (2.2) agree with the ones considered in the study of Dirichlet boundary value problems, and which are just the reversed ones to those usually considered in the study of second order initial value problems (see [11, 14]). This is due to the fact that we need to ensure the solvability of some regular Dirichlet problems in the proof of Theorem 2.2.

According to Definition 2.1, solutions might have discontinuous first derivatives at $x = 0$. This is not satisfactory for the motivating Problem (1.1) in its context, because the authors of [7] arrived at (1.1) after a change of variables in a related problem which consisted, roughly speaking, on turning an asymptotic condition at $+\infty$ into an initial condition at 0^+. Therefore the interesting point in [7] was really to find solutions to Problem (1.1) satisfying

$$\lim_{x \to 0^+} y'(x) = 0,$$

or, equivalently, to find solutions of (1.1) which are continuously differentiable on $[0, L)$.

Next corollary provides us with sufficient conditions for filling that gap.

Corollary 2.1 *In the conditions of Theorem 2.2, all solutions to (2.2) between* α *and* β *are continuously differentiable on* $[0, L)$ *provided that*

$$\text{either } f(x, y) \geq 0 \text{ whenever } \alpha(x) \leq y \leq \beta(x) \ (x \in I = (0, L]) \tag{2.9}$$

$$\text{or } f(x, y) \leq 0 \text{ whenever } \alpha(x) \leq y \leq \beta(x) \ (x \in I = (0, L]). \tag{2.10}$$

Proof. Either one of conditions (2.9) and (2.10) implies that solutions $y = y(x)$ of (2.2) between α and β have a monotone first derivative, so the limit $\lim_{x \to 0^+} y'(x) = y'(0^+)$ exists and, by virtue of the Darboux Theorem, it can only be $y'(0^+) = y'(0)$. □

3 Study of the motivating problem

We focus our attention now on Problem (1.1), so we consider Problem (2.2) with $y_1 = 0$ and

$$f(x,y) = \frac{\sqrt[3]{y}}{4\sqrt[3]{x^8}} - \frac{1}{6\sqrt[3]{x^5 y}}, \quad (x,y) \in (0,+\infty) \times (0,+\infty).$$

A useful property of f is that it is increasing with respect to the y variable. We will exploit this fact over and over in this section.

3.1 Existence of infinitely many solutions via explicit lower and upper solutions

Notice that $f(x,(2x/3)^{3/2}) = 0$ for all $x \in (0,+\infty)$, so $\alpha(x) = (2x/3)^{3/2}$ is a lower solution to (1.1) on $[0,+\infty)$. On the other hand, one can check that an upper solution for $x > 0$ near zero is given by $\beta(x) = x^{3/2}$. Specifically, we compute for $x > 0$

$$\beta''(x) = \frac{3}{4\sqrt{x}} < \frac{1}{12x^{13/6}} = f(x,\beta(x)) \text{ if and only if } 0 < x < 3^{-6/5},$$

and $3^{-6/5} = 0.26758052058674\ldots$. In particular, Theorem 2.2 applies on $[0,1/4]$. Here and henceforth we replace $3^{-6/5}$ by $1/4$ for simplicity in subsequent computations.

Furthermore, $f(x,y) \geq f(x,\alpha(x)) = 0$ whenever $y \geq \alpha(x)$, so Problem (1.1) has infinitely many continuously differentiable solutions between α and β on $[0,1/4]$, by virtue of Corollary 2.1.

To sum up, notice that we have proven the following theorem.

Theorem 3.1 *A lower and an upper solution for Problem (1.1) on the interval* $[0,1/4]$ *are given, respectively, by*

$$\alpha(x) = (2x/3)^{3/2} \quad \textit{and} \quad \beta(x) = x^{3/2} \quad \textit{for all } x \in [0,1/4]. \tag{3.11}$$

Therefore, for each $s \in [\alpha(1/4),\beta(1/4)] = [0.06804138174397\ldots, 0.125]$ *Problem (1.1) has a continuously differentiable solution* $y : [0,1/4] \to \mathbb{R}$ *such* *that* $y(1/4) = s$ *and*

$$(2x/3)^{3/2} \leq y(x) \leq x^{3/2} \quad \textit{for all } x \in [0, 1/4].$$

3.2 Complementary information about the set of solutions of Problem (1.1)

We complement the information given in Secton 3.1 by studying the global behavior of solutions to (1.1). We define the set of solutions

$$\mathcal{A} = \{y : [0, L_y) \to \mathbb{R} \text{ such that } y \text{ is a noncontinuable solution of } (1.1)\}, \quad (3.12)$$

where $L_y > 0$ might be equal to $+\infty$.

First we will show that each solution in $\mathcal{A}$ is univocally determined by its intersection with the graph of $\alpha(x) = (2x/3)^{3/2}$, $x > 0$. Indeed, we have the following result.

Theorem 3.2 *The mapping* $\mathcal{T} : \mathcal{A} \longrightarrow (0, \infty)$ *defined by*

$$\mathcal{T}y := x \in (0, L_y) \quad \textit{if } y(x) = \alpha(x)$$

is well–defined and bijective.

We will also show that solutions in $\mathcal{A}$ can be divided into two classes, one of them corresponding to solutions of (1.1) which tend to zero as x tends to some finite L_y (then defined only up to L_y), and another class of globally defined (i.e., defined up to $+\infty$) increasing solutions. More precisely, we have the following theorem which is illustrated in Figure 1.

Theorem 3.3 *There exists* $s_0 > 0$ *such that the following assertions hold:*

1. *If* $y \in \mathcal{A}$ *and* $\mathcal{T}y = s$ *for some* $s \in (0, s_0)$*, then*

$$L_y < +\infty \textit{ and } \lim_{x \to L_y^-} y(x) = 0.$$

2. *If* $y \in \mathcal{A}$ *and* $\mathcal{T}y = s$ *for some* $s \geq s_0$*, then*

$$L_y = +\infty \textit{ and } y \textit{ is increasing on } [0, +\infty).$$

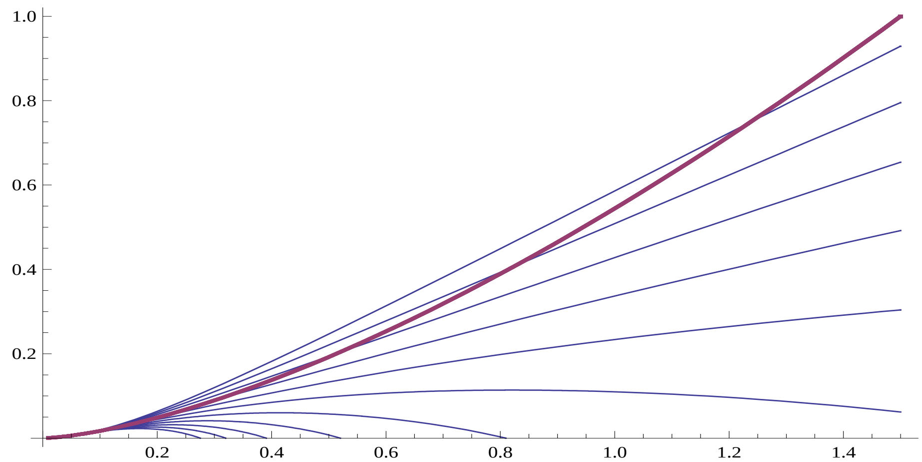

Figure 1: Some solutions of Problem (1.1) and their intersection with α, which is drawn thicker.

3.2.1 Proof of Theorem 3.2

$\mathcal{T} : \mathcal{A} \to (0, \infty)$ **is well–defined.** We start with the following lemma:

LEMMA 3.1 *Let $y \in \mathcal{A}$. If there exists $x_0 \in (0, L_y)$ such that $y(x_0) = \alpha(x_0)$ and $y'(x_0) \leq \alpha'(x_0)$ then*

$$y(x) < \alpha'(x_0)(x - x_0) + \alpha(x_0) =: l_{x_0}(x) < \alpha(x) \text{ for all } x \in (x_0, L_y).$$

Proof. Notice that the graph of the function l_{x_0} defined in the statement is nothing but the tangent to the graph of α at the point $(x_0, \alpha(x_0))$, which is below α because α is convex.

The assumptions $y(x_0) = \alpha(x_0)$, $y'(x_0) \leq \alpha'(x_0)$, and $y''(x_0) = f(x_0, \alpha(x_0)) = 0 < \alpha''(x_0)$, imply that $y < \alpha$ on $(x_0, x_0 + \delta)$ for some $\delta > 0$, and therefore $y''(x) < 0$ on $(x_0, x_0 + \delta)$ because f is negative below α. This implies that $y(x) < l_{x_0}(x)$ for all $x \in (x_0, x_0 + \delta)$.

We use a contradiction argument to prove our lemma: assume that there exists $x_1 > x_0$ such that

$$y(x) < l_{x_0}(x) \text{ for all } x \in (x_0, x_1) \text{ and } y(x_1) = l_{x_0}(x_1).$$

Then $y''(x) < 0$ for all $x \in (x_0, x_1)$ which, together with $y'(x_0) \le l'_{x_0}(x_0)$ and $y(x_0) = l_{x_0}(x_0)$, yield $y(x_1) < l_{x_0}(x_1)$, a contradiction. $\square$

Now we show that solutions of (1.1) are greater than the lower solution α near zero and they eventually go below it. Thus, in particular, solutions do not blow–up (this is a consequence of the sublinearity of the right–hand side).

PROPOSITION 3.1 *For every $y \in \mathcal{A}$ there exists $x_y \in (0, L_y)$ such that $y > \alpha$ on $(0, x_y)$ and $y < \alpha$ on (x_y, L_y). Therefore, we can define $\mathcal{T}y = x_y$ and, in particular, y is convex on $[0, x_y]$ and concave on $[x_y, L_y)$.*

Proof. We divide the proof into steps for better readability.
Step 1. There exists a decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers which tends to zero and $y(x_n) > \alpha(x_n)$ for all $n \in \mathbb{N}$. Reasoning by contradiction, assume there exists $\delta > 0$ such that $y \le \alpha$ on $[0, \delta]$. Hence $y'' \le 0$ on $[0, \delta]$, and therefore $y(x) \le 0$ for $x \in [0, \delta]$, which is impossible.

Step 2. There exists $\delta > 0$ such that $y > \alpha$ on $(0, \delta)$. If not, there exists a decreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ which tends to zero and $y(s_n) \le \alpha(s_n)$. For each $n \in \mathbb{N}$ we can find $m \in \mathbb{N}$, $m \ge n$, such that $x_m < s_n$ (x_m as in Step 1), and the Bolzano's theorem yields the existence of some $z_n \in (x_m, s_n]$ such that $y > \alpha$ on (x_m, z_n) and $y(z_n) = \alpha(z_n)$. In particular, $y'(z_n) \le \alpha'(z_n)$, and then Lemma 3.1 ensures

$$y(x) < \alpha'(z_n)(x - z_n) + \alpha(z_n) \text{ for all } x \in (z_n, L_y). \tag{3.13}$$

Now let $x \in (0, L_y)$ be fixed and notice that (3.13) holds for all sufficiently large values of $n \in \mathbb{N}$, so letting n tend to infinity in (3.13) we obtain $y(x) \le 0$, a contradiction.

Finally, we prove that x_y in the conditions of the statement exists. By Lemma 3.1 and the information obtained in Step 2, it suffices to prove that there exists $x \in (0, L_y)$ such that $y(x) \le \alpha(x)$. Assume, on the contrary, that $y > \alpha$ on $(0, L_y)$. Then y'' is positive on $(0, L_y)$ and both y and y' are increasing and positive on $(0, L_y)$.

Let us prove that $L_y = +\infty$. If not, we have two possibilities: either

$$\lim_{x \to L_y^-} y(x) = y_\infty < +\infty \text{ and } \lim_{x \to L_y^-} y'(x) = +\infty, \tag{3.14}$$

or

$$\lim_{x \to L_y^-} y(x) = +\infty, \tag{3.15}$$

since otherwise the solution can be extended past L_y.

Next we show that neither (3.14) nor (3.15) are possible. First, if (3.14) holds then for a fixed $x_0 \in (0, L_y)$ and every $x \in (x_0, L_y)$ we have

$$y'(x) = y'(x_0) + \int_{x_0}^{x} f(s, y(s))ds \le y'(x_0) + \int_{x_0}^{L_y} f(s, y_\infty)ds,$$

hence the second identity in (3.14) cannot be fulfilled.

Second, we assume that (3.15) is satisfied. Let $x_0 \in (0, L_y)$ be such that $y(x_0) = 1$ and $y(x) \ge 1$ for all $x \in (x_0, L_y)$, so that

$$y''(x) \le \frac{y(x)}{4x_0^{8/3}} \text{ for all } x \in (x_0, L_y). \tag{3.16}$$

Multiplying in (3.16) by y' and integrating between x_0 and $x \in (x_0, L_y)$ we arrive at

$$\begin{aligned} {y'}^2(x) &\le \frac{1}{4x_0^{8/3}} y^2(x) - \frac{1}{4x_0^{8/3}} + {y'}^2(x_0) \\ &\le \max\left\{\frac{1}{4x_0^{8/3}}, {y'}^2(x_0)\right\}(y^2(x) + 1) \le \max\left\{\frac{1}{4x_0^{8/3}}, {y'}^2(x_0)\right\} 2y^2(x), \end{aligned}$$

so for $x \in (x_0, L_y)$ we have

$$y'(x) \le c\, y(x) \qquad \left(c = \sqrt{2 \max\left\{\frac{1}{4x_0^{8/3}}, {y'}^2(x_0)\right\}}\right),$$

which implies that

$$y(x) \le y(x_0)\mathrm{e}^{c(x - x_0)} \quad \text{for } x \in (x_0, L_y),$$

a contradiction with (3.15). The proof that $L_y = +\infty$ is complete.

Notice that y' cannot be bounded above on $(0,+\infty)$, because in that case y would be sublinear and it would eventually go below α, which has a superlinear growth. Therefore $y'(x)$ tends to $+\infty$ as x tends to $+\infty$. This enables us to use the L'Hôpital rule in the following computations:

$$\begin{aligned}\lim_{x\to+\infty} y''(x)x^{1/2} &= \lim_{x\to+\infty}\frac{y^{1/3}(x)}{4x^{13/6}} = \lim_{x\to+\infty}\frac{y'(x)}{26x^{7/6}y^{2/3}(x)}\\ &= \lim_{x\to+\infty}\frac{f(x,y(x))}{\frac{91}{3}x^{1/6}y^{2/3}(x)+\frac{52}{3}x^{7/6}y^{-1/3}(x)y'(x)}\\ &= \lim_{x\to+\infty}\frac{y^{1/3}(x)}{4x^{8/3}\left(\frac{91}{3}x^{1/6}y^{2/3}(x)+\frac{52}{3}x^{7/6}y^{-1/3}(x)y'(x)\right)}\\ &= \lim_{x\to+\infty}\frac{y^{2/3}(x)}{\frac{364}{3}x^{17/6}y(x)+\frac{208}{3}x^{23/6}y'(x)} = 0,\end{aligned}$$

and thus, once again by virtue of the L'Hôpital rule, we obtain

$$\lim_{x\to+\infty}\frac{y(x)}{\alpha(x)} = \lim_{x\to+\infty}\frac{y''(x)}{\alpha''(x)} = 0,$$

a contradiction with $y > \alpha$ on $(0,+\infty)$. □

Next we prove that all solutions of (1.1) go below β near zero.

Proposition 3.2 *If $y \in \mathcal{A}$ then there exists $\varepsilon_y \in (0,1/4)$ such that $y(x) < \beta(x) = x^{3/2}$ for $x \in (0,\varepsilon_y)$.*

Proof. First, notice that it is impossible to have $y \ge \beta$ on $[0,\varepsilon]$ for some $\varepsilon > 0$, because in that case we would have

$$y''(x) = f(x,y(x)) \ge f(x,\beta(x)) = \frac{1}{12x^{13/6}} \quad \text{for all } x \in (0,\varepsilon],$$

so y'' would not be integrable on $(0,\varepsilon]$, a contradiction. Therefore, there exists a decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ that converges to zero such that $y(x_n) < \beta(x_n)$ for all $n \in \mathbb{N}$ and $x_1 \le 1/4$.

Now we use a contradiction argument: suppose that we can find $s \in (0,x_1)$ such that $y(s) \ge \beta(s)$. Let $n \in \mathbb{N}$ be so large that $0 < x_n < s < x_1$. By virtue of the Bolzano's theorem, there exist $a \in (x_n,s]$ such that $y < \beta$ on (x_n,a) and $y(a) = \beta(a)$. Notice that $y'(a) \ge \beta'(a)$ and

$$y''(a) = f(a,y(a)) = f(a,\beta(a)) > \beta''(a),$$

hence there exists $\delta > 0$ such that $y > \beta$ on $(a, a + \delta)$. Moreover, we have $a < x_1$, so we can find $b \in (a, x_1)$ such that

$$y(x) > \beta(x) \text{ for all } x \in (a, b) \text{ and } y(b) = \beta(b).$$

Now for $x \in (a, b)$ we have

$$y''(x) = f(x, y(x)) > f(x, \beta(x)) > \beta''(x),$$

and therefore $y(b) > \beta(b)$, a contradiction. □

<u>$\mathcal{T} : \mathcal{A} \to (0, \infty)$ **is one–to–one.**</u> As a consequence of the following Proposition we deduce that each point in the graph of α is crossed by, at most, one solution.

PROPOSITION 3.3 *Let y_1, $y_2 \in \mathcal{A}$. If $y_1(x_0) \leq y_2(x_0)$ for some $x_0 > 0$ then $y_1 \leq y_2$ on the intersection of their domains.*

Proof. Let y_i $(i = 1, 2)$ be two solutions in the conditions of the statement for some $x_0 > 0$, and assume that the lemma is not true, i.e., assume that there exists $x_1 > 0$ such that $y_1(x_1) > y_2(x_1)$. Two cases are possible: either $x_1 < x_0$ or $x_0 < x_1$.

If $x_1 < x_0$ then there exist $a, b \in \mathbb{R}$ such that $0 \leq a < x_1 < b \leq x_0$,

$$y_1 - y_2 > 0 \text{ on } (a, b), \text{ and } (y_1 - y_2)(a) = 0 = (y_1 - y_2)(b). \tag{3.17}$$

Since $f(x, y)$ is increasing with respect to y, we have for $x \in (a, b)$ that

$$(y_1 - y_2)''(x) = f(x, y_1(x)) - f(x, y_2(x)) > 0,$$

so $y_1 - y_2$ is convex on $[a, b]$, a contradiction with (3.17).

If $x_0 < x_1$ then there exists $x_2 \in [x_0, x_1)$ such that $y_1(x_2) = y_2(x_2)$ and $y_1 > y_2$ on $(x_2, x_1]$. Notice that $y_1'(x_2) = y_2'(x_2)$ implies $y_1 = y_2$ on $[x_2, x_2 + \delta]$ for some $\delta > 0$ by virtue of the Lipschitz theorem, a contradiction with $y_1 > y_2$ on $(x_2, x_1]$, therefore we have $y_1'(x_2) > y_2'(x_2)$. Hence we can find $a \in \mathbb{R}$ such that $0 \leq a < x_2$ and

$$y_2 - y_1 > 0 \text{ on } (a, x_2), \quad (y_2 - y_1)(a) = 0 = (y_2 - y_1)(x_2),$$

a contradiction with $(y_2 - y_1)'' > 0$ on (a, x_2). $\square$

$\mathcal{T} : \mathcal{A} \to (0, \infty)$ **is surjective.** Now we prove that every point in the graph of α is actually crossed by one solution of (1.1).

PROPOSITION 3.4 *For each $x_0 > 0$ there exists $y \in \mathcal{A}$ such that $y(x_0) = \alpha(x_0)$.*

Proof. We define the set $\mathcal{S}$ of all those $s > 0$ such that there is y_s a solution to (1.1) satisfying $y_s(s) = \alpha(s)$. Notice that at least $(0, 1/4] \subset \mathcal{S}$, as we deduce by means of Theorem 2.2, the lower solution α and the upper solution β on each interval $[0, s]$, $s \in (0, 1/4]$.

Next we prove that $\mathcal{S}$ is an open interval.

First, we show that $\mathcal{S}$ is connected. Let $s_i \in \mathcal{S}$, $i = 1, 2$, be such that $s_1 < s_2$; we are going to prove that $[s_1, s_2] \subset \mathcal{S}$. To do so, we fix an arbitrary $s \in (s_1, s_2)$ and we take y_2, a solution of (1.1) such that $y_2(s_2) = \alpha(s_2)$. Plainly, α and y_2 are, respectively, lower and upper solutions of (1.1) on $[0, s]$. Moreover, Proposition 3.1 guarantees that $y_2 \geq \alpha$ on $[0, s]$, so Theorem 2.2 ensures the existence of a solution of (1.1), say $y = y(x)$, such that $\alpha \leq y \leq y_2$ on $[0, s]$ and $y(s) = \alpha(s)$. Hence $s \in \mathcal{S}$.

Second, we show that $\mathcal{S}$ is open. Let $s_0 \in \mathcal{S}$ be fixed, and let $y_0 : [0, s_0 + \delta] \to \mathbb{R}$ ($\delta > 0$) be the solution of (1.1) such that $y_0(s_0) = \alpha(s_0)$. Proposition 3.2 ensures that we can find $\varepsilon_0 \in (0, 1/4)$ such that

$$y_0(x) < \beta(x) \text{ for all } x \in (0, \varepsilon_0),$$

which implies that $y_0'(x_0) < \beta'(x_0)$ for some $x_0 \in (0, \varepsilon)$. Now we define a function

$$\hat{\beta}(x) = \begin{cases} \beta(x) & \text{if } 0 \leq x \leq x_0, \\ \\ y_0(x) + \beta(x_0) - y_0(x_0) & \text{if } x_0 < x \leq s_0 + \delta, \end{cases}$$

which is an upper solution of (1.1) on $[0, s_0 + \delta]$ because $\hat{\beta} > y_0$, so for $x \in (x_0, s_0 + \delta]$ we have

$$\hat{\beta}''(x) = y_0''(x) = f(x, y_0(x)) < f(x, \hat{\beta}(x)),$$

and $\hat\beta'_-(x_0) = \beta'(x_0) > y_0'(x_0) = \hat\beta'_+(x_0)$.

Now Theorem 2.2, the lower solution α and the upper solution $\hat\beta$ on $[0, s_0+\hat\delta]$, where $\hat\delta \in (0,\delta)$ is chosen so that $\hat\beta(s_0+\hat\delta) \geq \alpha(s_0+\hat\delta)$, imply that $(0, s_0+\hat\delta] \subset \mathcal{S}$, so s_0 is an interior point of $\mathcal{S}$. Since x_0 was arbitrarily fixed, we conclude that $\mathcal{S}$ is open. Therefore, $\mathcal{S}$ is an open interval because it is open and connected, so there exists $s_\infty \in (0, +\infty]$ such that $\mathcal{S} = (0, s_\infty)$ and then it only remains to prove that $s_\infty = +\infty$.

Assume that $s_\infty < +\infty$, let $\{s_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers which converges to s_∞ and let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of solutions of (1.1) such that $y(s_n) = \alpha(s_n)$ for all $n \in \mathbb{N}$. In order to deal only with positive derivatives, we redefine y_n to be α on $[s_n, s_\infty]$.

Now for all $n \in \mathbb{N}$ and all $x \in [0, s_\infty]$, $x \neq s_n$, we have $0 \leq y_n'(x) \leq \alpha'(s_\infty) =: C$, and then, up to a subsequence, $\{y_n\}_{n\in\mathbb{N}}$ converges uniformly on $[0, s_\infty]$ to some continuous function $y_\infty : [0, s_\infty] \to [0, +\infty)$.

Notice that $y_n \leq y_{n+1}$ on $[0, s_\infty]$ for all $n \in \mathbb{N}$, as one can deduce by means of the ideas in the proof of Proposition 3.3, and thus the monotone convergence theorem ensures for all $x \in [0, s_\infty)$ that

$$C \geq \lim_{n\to\infty} y_n'(x) = \lim_{n\to\infty} \int_0^x f(r, y_n(r))dr = \int_0^x f(r, y_\infty(r))dr.$$

In particular, the composition $f(\cdot, y_\infty(\cdot)) \in \mathrm{L}^1(0, s_\infty)$.

The dominated convergence theorem yields for $x \in [0, s_\infty)$ that

$$y_\infty(x) = \lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} \int_0^x y_n'(s)ds = \int_0^x \int_0^s f(r, y_\infty(r))drds,$$

so y_∞ is a solution of (1.1) such that $y_\infty(s_\infty) = \alpha(s_\infty)$. This means that $s_\infty \in \mathcal{S}$, so $\mathcal{S}$ is not open, a contradiction. ◻

3.2.2 Proof of Theorem 3.3

Since solutions are (strictly) concave when their graphs go below that of α, we only have two possibilities for $y \in \mathcal{A}$ with $\mathcal{T}y = x_y$: either $y' > 0$ on $[x_y, L_y)$, which implies that $L_y = +\infty$, or y' vanishes at some point in (x_y, L_y), which implies that y tends to zero as x tends to $L_y < +\infty$.

We start proving the following claim.

Claim 1– There exists $x_0 \in (0,1)$ *such that if* $y \in \mathcal{A}$ *with* $\mathcal{T}y = x_0$ *then*

$$L_y < +\infty \text{ and } \lim_{x \to L_y^-} y(x) = 0.$$

Assume that the claim is false, i.e., for every $x_0 \in (0,1)$ the solution y of (1.1) such that $y(x_0) = \alpha(x_0)$ is defined on $[0,+\infty)$.

By Lemma 3.1, we have

$$y(x) < l_{x_0}(x) < \alpha(1) = (2/3)^{3/2} \quad \text{for } x \in (1, x_1),$$

where $x_1 > 1$ solves $l_{x_0}(x_1) = \alpha(1)$. In particular, x_1 depends linearly on x_0 and x_1 tends to $+\infty$ as x_0 tends to zero.

We compute

$$\begin{aligned} y'(x_1) &= y'(1) + \int_1^{x_1} f(s,y(s))ds \le y'(x_0) + \int_1^{x_1} f(s,\alpha(1))ds \\ &\le \alpha'(x_0) - \frac{1}{10}\sqrt{\frac{3}{2}}\frac{1}{x_1^{5/3}} + \frac{1}{4}\sqrt{\frac{3}{2}}\frac{1}{x_1^{2/3}} - \frac{3}{20}\sqrt{\frac{3}{2}} =: g(x_0). \end{aligned}$$

Notice that

$$\lim_{x_0 \to 0^+} g(x_0) = -\frac{3}{20}\sqrt{\frac{3}{2}} < 0,$$

so there exists $\delta \in (0,1)$ such that $g(x_0) < 0$ provided that $x_0 \in (0,\delta)$, and therefore, the corresponding solution y has negative derivative at x_1. Since y is concave on the right of x_0, we conclude that y is defined on some finite interval $[0, L_y)$ and tends to zero as x tends to L_y, a contradiction. The claim is proven.

Next we show that solutions defined on the whole of $[0,+\infty)$ exist:

Claim 2– There exists $x_0 > 1$ *such that if* $y \in \mathcal{A}$ *and* $\mathcal{T}y = x_0$ *then* $L_y = +\infty$.

First, we extend linearly the definition of the upper solution $\beta(x) = x^{3/2}$, $x \in [0,1/4]$, as follows: let $\hat{\beta} = \beta$ on $[0,1/4]$ and for $x \ge 1/4$ define

$$\hat{\beta}(x) = \frac{3}{4}c\left(x - \frac{1}{4}\right) + \frac{1}{8} \quad \text{where } c < 1.$$

According to Definition 2.1, $\hat{\beta}$ is an upper solution on the interval $[0, x_0]$, where $\hat{\beta}(x_0) = \alpha(x_0)$.

Notice that $11/16 > (2/3)^{3/2}$, so for $c < 1$ sufficiently close to 1 we have

$$\hat{\beta}(1) = \frac{11c}{16} > \left(\frac{2}{3}\right)^{3/2} = \alpha(1).$$

This choice of c implies that $x_0 > 1$ and that $\hat{\beta}(x_0) > \alpha(1)$.

By virtue of Theorem 2.2, with the lower solution α and the upper solution $\hat{\beta}$ on $[0, x_0]$, we conclude that there exists a solution of Problem (1.1), $y : [0, L_y) \to \mathbb{R}$ with $L_y > x_0$, and such that

$$\alpha < y < \hat{\beta} \quad \text{on } (0, x_0) \text{ and } \alpha(x_0) = y(x_0) = \hat{\beta}(x_0).$$

In particular, $y'(x_0) \geq \hat{\beta}'(x_0) = \beta'(1/4) = 3c/4$ and $y(x) > \alpha(1)$ on $(x_0, x_0 + \varepsilon)$ for some $\varepsilon > 0$.

We are going to prove that $y > \alpha(1)$ on (x_0, L_y), which yields $L_y = \infty$, since otherwise the solution y could be extended on the right of L_y.

Suppose, on the contrary, that there exists $x_1 > x_0$ such that $y(x) > \alpha(1)$ for all $x \in [x_0, x_1)$ and $y(x_1) = \alpha(1)$. We then have for $x \in (x_0, x_1)$

$$y''(x) = f(x, y(x)) > f(x, \alpha(1)) = \frac{1}{2\sqrt{6}x^{8/3}} - \frac{1}{2\sqrt{6}x^{5/3}},$$

whence

$$\begin{aligned} y'(x) &> y'(x_0) + \frac{1}{2\sqrt{6}} \left(\int_{x_0}^{x} s^{-8/3} ds - \int_{x_0}^{x} s^{-5/3} ds \right) \\ &= \frac{3c}{4} + \frac{1}{2\sqrt{6}} \left(-\frac{3}{5}x^{-5/3} + \frac{3}{5}x_0^{-5/3} + \frac{3}{2}x^{-2/3} - \frac{3}{2}x_0^{-2/3} \right) \\ &> \frac{1}{2\sqrt{6}} \left(\frac{3}{2} - \frac{3}{5} \right) x^{-5/3} + \frac{3c}{4} - \frac{3}{4\sqrt{6}} > 0, \quad \text{if } c \text{ is close to } 1. \end{aligned}$$

Therefore y is increasing on (x_0, x_1), a contradiction with $y(x_1) = \alpha(1) < y(x_0)$. Claim 2 is proven.

Now we define $\mathcal{S}$ as the set of all those real numbers $s > 0$ such that $\mathcal{T}y = s$ implies $L_y < +\infty$ and y tends to zero as x tends to L_y. Claim 1 shows that $\mathcal{S}$ is not empty.

Next we prove that $s_0 = \sup \mathcal{S}$ exists and satisfies all the properties in the statement. We divide the proof into three steps for better readability.

Step 1– $\mathcal{S}$ is bounded above. Let $x_0 > 1$ be as in Claim 2 and let y_0 be the solution of (1.1) which fulfills $y_0(x_0) = \alpha(x_0)$. We are going to prove that x_0 is an upper bound for $\mathcal{S}$. Reasoning by contradiction, assume that there exists $s \in \mathcal{S}$ such that $s > x_0$, and let $y \in \mathcal{A}$ such that $\mathcal{T}y = s$. Since $y(x_0) > \alpha(x_0) = y_0(x_0)$, Proposition 3.3 yields $y > y_0 > 0$ on $[0, L_y)$, hence

$$\lim_{x \to L_y^-} y_0(x) = 0,$$

and y_0'' does not exist at L_y, a contradiction.

Step 2– If $y \in \mathcal{A}$ and $\mathcal{T}y = s < s_0$ then $s \in \mathcal{S}$, i.e.,

$$L_y < +\infty \quad \textit{and} \quad \lim_{x \to L_y^-} y(x) = 0.$$

Let $y \in \mathcal{A}$ be such that $\mathcal{T}y = s$ for some $s < s_0$, and fix $s^* \in \mathcal{S}$, $s^* > s$. Let $y^* : [0, L_{y^*}) \to \mathbb{R}$ be the solution of (1.1) such that $y^*(s^*) = \alpha(s^*)$. We have $y^*(s) > \alpha(s) = y(s)$ so Proposition 3.3 ensures $y^* \geq y$ on the intersection of their domains, hence $L_y \leq L_{y^*}$ and

$$\lim_{x \to L_y^-} y(x) = 0,$$

since otherwise we would have

$$\lim_{x \to L_{y^*-}} y(x) = 0,$$

and y'' would not exist at L_{y^*}.

Step 3– If $y \in \mathcal{A}$ and $\mathcal{T}y = s \geq s_0$ then $L_y = +\infty$. The result is an immediate consequence of the definition of supremum in case $s > s_0$. Now we focus our attention in the case $s = s_0$.

Reasoning by contradiction once again, we assume that $\mathcal{T}y = s_0$ and $L_y < +\infty$. We construct an upper solution above y as in the proof of Proposition 3.4: there exists $\delta \in (0, 1/4)$ such that the function

$$\hat{\beta}(x) = \begin{cases} \beta(x) & \text{if } 0 \leq x \leq \delta, \\ y(x) + \beta(\delta) - y(\delta) & \text{if } \delta < x < L_y, \end{cases}$$

is an upper solution of (1.1) on $[0, L_y)$ with $\hat{\beta}'_-(\delta) > \hat{\beta}'_+(\delta)$. We use Theorem 2.2 with this upper solution and the lower solution y to deduce the existence of a solution $\hat{y} > y$ on $(0, L_y)$ such that $\hat{y}(L_y) < y(s_0)$. Since $y(s_0) < \hat{y}(s_0)$, we deduce that $\hat{y}'$ is negative near L_y, so $\hat{y}$ cannot be extended to $[0, +\infty)$. Moreover, $\hat{y}(r) = \alpha(r)$ for some $r > s_0$, hence $r \in \mathcal{S}$ and $r > s_0$, a contradiction.

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