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The Brouwer fixed point theorem and periodic solutions of differential equations

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Abstract. The Brouwer fixed point theorem is a key ingredient in the proof that a periodic differential equation has a periodic solution in a set that satisfies a suitable tangency condition on its boundary. The main goal of this note is to show that both results are in fact equivalent.

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1. Introduction

The Brouwer fixed point theorem [2, Satz 4] has a simple statement, a very interesting history and a long list of rich applications, see [1,5,17,21].

Brouwer fixed point theorem. If $C \subset \mathbb{R}^n$ is a nonempty, closed, bounded and convex set and $g : C \to C$ is continuous, then there exists $x \in C$ such that $g(x) = x$.

The first application of the Brouwer fixed point theorem to the existence of periodic solutions of differential equations seems to be the one given in the pioneering work by Lefschetz [14]. He considered the Liénard equation

$$x'' + a'(x)x' + b(x) = e(t), \tag{1}$$

where $e(t)$ is $T-$periodic and gave sufficient conditions in order that an ellipse S is an invariant set for the Poincaré map $\mathcal{P}$ of the equivalent planar system

$$\begin{cases} x' = -a(x) + y, \\ y' = -b(x) + e(t). \end{cases} \tag{2}$$

We recall that $\mathcal{P}$ is defined as

$$\mathcal{P}(x_0, y_0) := (x(T, 0, (x_0, y_0)), y(T, 0, (x_0, y_0))),$$

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where $(x(\cdot,0,(x_0,y_0)),y(\cdot,0,(x_0,y_0)))$ is the solution of (2) with initial condition (x_0,y_0) at time $t=0$. It is an easy task to show that the fixed points of $\mathcal{P}$ are precisely the initial conditions of the $T-$periodic solutions of (1). A direct application of the Brouwer fixed point theorem to the continuous map $\mathcal{P}: S \to S$ yields the desired $T-$periodic solution.

Lefschetz's result was quickly followed by a generalization by Levinson, [15], and good accounts of the application of Brouwer fixed point theorem to the search of periodic solutions can be found in the monographs [22,23].

A major step in that direction was given by Krasnosels'kii in [10, Theorem 3.2], to show that if $f:\mathbb{R}\times\Omega\to\mathbb{R}^n$ is a continuous function, $T-$periodic in t (i.e., $f(t,x)=f(t+T,x)$ for each $(t,x)\in\mathbb{R}\times\Omega$), Ω is a "canonical" closed, convex and bounded region, and f satisfies a suitable tangency condition on $\partial\Omega$, then the differential equation

$$x' = f(t,x), \tag{3}$$

has at least one $T-$periodic solution that lies in Ω. A brief outline of Krasnosels'kii's approach goes as follows: supposing in addition that f is Lipschitz continuous, which guarantees the uniqueness and continuous dependence on the initial conditions of the solutions of (3), the assumed tangency condition implies that every point $x\in\Omega$ remains in Ω when following a trajectory of the $T-$periodic equation (3). This means that its Poincaré map is well–defined and maps continuously Ω into itself, so the Brouwer fixed point theorem provides a fixed point which yields the initial condition of a $T-$periodic solution of (3). Now, the general result follows by uniformly approximating the continuous function f by a sequence f_n of Lipschitz continuous functions satisfying the tangency condition and by using a compactness argument.

The Krasnosels'kii theorem is a special case of a more general result on the existence of periodic solutions lying in an open convex neighborhood of 0, first proved by Gustafson and Schmitt [7], in the more general situation of delay-differential equations, using Leray-Schauder degree. A simpler proof, based upon coincidence degree was given in [16, Corollary 3.1] as well as its link with Krasnosel'skii's theorem in [16, Corollary 3.2]. A variant of the Gustafson-Schmitt's result states as follows.

Theorem 1.1. *Assume that $C\subset\mathbb{R}^n$ is a nonempty, closed, bounded and convex set, $f:[0,T]\times C\to\mathbb{R}^n$ is continuous, and that, for each outer normal field $\nu:\partial C\to S^{n-1}$, the condition*

$$\langle \nu(x), f(t,x)\rangle \le 0 \ \textit{ for all } \ (t,x)\in[0,T]\times\partial C \tag{4}$$

holds. Then the equation (3) has at least one solution x such that $x(0)=x(T)$ and $x(t)\in C$ for all $t\in[0,T]$.

Notice that when $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is T-periodic with respect to t, the solutions on $[0,T]$ verifying the T-periodic boundary condition $x(0)=x(T)$ can be extended by periodicity to T-periodic solutions.

The main result of this note is to prove the following equivalence theorem.

Theorem 1.2. *Theorem 1.1 is equivalent to the Brouwer fixed point theorem.*

To this aim, we first give a proof of Theorem 1.1 based upon the Brouwer fixed point theorem in a closed ball. To avoid the introduction of locally Lipschitzian approximations of f requested by the use of the Poincaré map, we introduce instead a modified problem whose T-periodic solutions are the ones of the original problem in C, obtain approximate solutions of the corresponding Cauchy problem, employ the Brouwer fixed point theorem to obtain approximate T-periodic solutions of the modified problem, and finally use the Ascoli-Arzelá theorem to extract a T-periodic solution of the modified, and hence of the original problem.

The modified problem, and its relation with the original one, involve some basic concepts of convex analysis. The approximate solutions are obtained by an approach to the Cauchy problem introduced in 1906 by de la Vallée Poussin in the first edition of his *Cours d'analyse infinitésimale* [4, Théorème fondamental, p. 130], rediscovered in 1928 by Tonelli [25] in the setting of nonlinear Volterra integral equations, and applied in 1947 by Stampacchia [24] to boundary value problems for nonlinear differential equations. This not so well known approach is applied to periodic problems in [5] and named Stampacchia's method. The idea consists in defining approximate solutions x_k $(k = 1, 2, \ldots)$ to the Cauchy problem

$$x'(t) = f(t, x(t)),\ x(0) = a$$

by solving the Cauchy problem for the retarded equations

$$\begin{aligned}
&x_k(t) = a \ \text{ for all } \ t \in [-k^{-1}T, 0],\\
&x_k'(t) = f(t, x_k(t - k^{-1}T)) \ \text{ for all } \ t \in (0, T],
\end{aligned}$$

step by step by quadratures on the subintervals $((j-1)k^{-1}T, jk^{-1}T]$ $(j = 1, \ldots, k)$ of length $k^{-1}T$.

To complete the proof of Theorem 1.2, it remains to show that the Brouwer fixed point theorem is a consequence of Theorem 1.1. If $g : C \to C$ is continuous, this is done by showing that the autonomous differential equation

$$x'(t) = g(x(t)) - x(t) \tag{5}$$

satisfies the assumptions of Theorem 1.1 for each $T > 0$. A sequence of T_k-periodic solutions of (5) with $T_k \to 0$ as $k \to \infty$ provides an equilibrium of (5) and hence a fixed point of g.

Despite of the impressive list of equivalent reformulations of Brouwer fixed point theorem in the literature (see for instance [1,5,12,13,17,21]), the equivalence between the Brouwer fixed point theorem and the existence of periodic solutions for some differential equation seems to have been unnoticed. Although Poincaré had used a topological statement equivalent to the Brouwer fixed point theorem (the so-called Poincaré-Miranda theorem) to study periodic solutions in celestial mechanics as early as 1883, and furthermore had shown how to reduce the existence of periodic solutions of differential systems to the fixed points of the Poincaré map, it is not known if Poincaré heard about the Brouwer fixed point theorem, published shortly before his dead. On the other hand, Brouwer never gave any application of his fixed point theorem, and one had to wait the beginning of the years 1940 for a proof of the equivalence between the Poincaré-Miranda and Brouwer theorems, and for the first use of the Brouwer fixed point theorem in finding periodic solutions.

Another application of the approach used in this paper is a very short proof using the Brouwer fixed point theorem of a slight improvement of a recent result of Fonda and Gidoni [6] on the existence of zeros of some mappings in convex sets (Theorem 5.1).

The paper is organized as follows. In Section 2 we collect the needed properties of the projection operator on a closed convex set in $\mathbb{R}^n$. The proof of Theorem 1.1 based upon the Stampacchia method and the Brouwer fixed point theorem is given in Section 3. In Section 4 we complete the proof of Theorem 1.2 by deducing the Brouwer fixed point theorem from Theorem 1.1. Finally, in Section 5, we present related results on the existence of zeros of nonlinear mappings, further remarks, and open problems.

2. Preliminaries on convex analysis

In this section we describe the main properties of the nonlinear projection operator onto closed convex sets and the related concepts of normal and tangent vectors to a convex set [9]. Through the paper $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the usual Euclidean inner product and norm, respectively.

Let $C \subset \mathbb{R}^n$ be a nonempty, closed and convex set and let $p_C : \mathbb{R}^n \to C$ be the orthogonal projection onto C defined by the formula

$$\|x - p_C(x)\| = \min_{y \in C} \|x - y\|.$$

Indeed, $p_C(x)$ is well-defined and also characterized as the unique element in C that satisfies the variational inequality

$$\langle x - p_C(x), y - p_C(x) \rangle \leq 0 \quad \text{for all } y \in C.$$

We point out that p_C satisfies the following properties :

i) $p_C(x) = x$ if and only if $x \in C$.
ii) $p_C(x) \in \partial C$ for each $x \in \mathbb{R}^n \setminus C$.
iii) $\|p_C(x) - p_C(y)\| \leq \|x - y\|$ for all $x, y \in \mathbb{R}^n$, that is, p_C is nonexpansive.

The normal and tangent cones to C at $x \in C$, denoted by $N_C(x)$ and $T_C(x)$, are defined respectively as

$$N_C(x) = \{u \in \mathbb{R}^n : \langle u, v - x \rangle \leq 0 \quad \text{for all } v \in C\},$$

$$T_C(x) = \{v \in \mathbb{R}^n : \langle v, w \rangle \leq 0 \quad \text{for all } w \in N_C(x)\}.$$

If $x \in \mathbb{R}^n \setminus C$ then $0 \neq x - p_C(x) \in N_C(p_C(x))$ and as consequence it is easy to show that for each $x \in \partial C$ there exists at least one unitary vector $\nu(x) \in N_C(x)$. We will call such an application

$$\nu : \partial C \to S^{n-1} \cap N_C(x)$$

a *normal outer field*. Notice that ν needs not to be neither unique nor continuous.

We need a less familiar result about p_C, which can be essentially found in [9], Chapters III and IV, but we give a proof for the reader's convenience. Define the function $\delta_C : \mathbb{R}^n \to \mathbb{R}$ by

$$\delta_C(x) = \frac{1}{2}\|x - p_C(x)\|^2 \ \text{ for all } \ x \in \mathbb{R}^n.$$

Lemma 2.1. *For each $x \in \mathbb{R}^n$, the function δ_C is differentiable and*

$$\nabla \delta_C(x) = x - p_C(x). \tag{6}$$

Proof. For all $x \in \mathbb{R}^n$ and all $h \in \mathbb{R}^n$, we have, by definition of p_C,

$$\delta_C(x) \le \frac{1}{2}\|x - p_C(x+h)\|^2 \ \text{ and } \ \delta_C(x+h) \le \frac{1}{2}\|x + h - p_C(x)\|^2.$$

Therefore,

$$\begin{aligned}
\delta_C(x+h) - \delta_C(x) &\ge \frac{1}{2}[\|x+h-p_C(x+h)\|^2 - \|x - p_C(x+h)\|^2] \\
&= \frac{1}{2}\langle 2x + h - 2p_C(x+h), h\rangle \\
&= \langle x - p_C(x), h\rangle + \|h\|^2 - \langle p_C(x+h) - p_C(x), h\rangle \\
&\ge \langle x - p_C(x), h\rangle
\end{aligned}$$

where we have used the non-expansive character of p_C. Furthermore,

$$\begin{aligned}
\delta_C(x+h) - \delta_C(x) &\le \frac{1}{2}[\|x+h-p_C(x)\|^2 - \|x - p_C(x)\|^2] \\
&= \frac{1}{2}\langle 2x + h - 2p_C(x), h\rangle = \langle x - p_C(x), h\rangle + \|h\|^2.
\end{aligned}$$

Hence, for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$

$$\langle x - p_C(x), h\rangle \le \delta_C(x+h) - \delta_C(x) \le \langle x - p_C(x), h\rangle + \|h\|^2.$$

□

3. Proof of Theorem 1.1

3.1. The modified problem

To study the first order periodic boundary value problem on $[0, T]$

$$x' = f(t, x), \ x(0) = x(T), \tag{7}$$

we introduce the modified problem

$$y'(t) = -y(t) + p_C(y(t)) + f(t, p_C(y(t))), \ t \in [0, T], \quad y(0) = y(T), \tag{8}$$

and define $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$h(t, y) := p_C(y) + f(t, p_C(y)). \tag{9}$$

Notice that h is continuous and bounded on $[0, T] \times \mathbb{R}^n$ and that the modified problem (8) reduces to the original one (7) on $[0, T] \times C$.

Using the variation of constants formula, any solution of the differential equation in (8) such that $y(0) = a \in \mathbb{R}^n$ satisfies the nonlinear integral equation

$$y(t) = e^{-t}a + \int_0^t e^{-(t-s)} h(s, y(s))\, ds \tag{10}$$

for all $t \geq 0$ for which it exists.

3.2. The Stampacchia method

The following Stampacchia method [5,24], based upon finite-dimensional techniques, consists in defining the sequence of approximate solutions $y_k(\cdot, a) : [-k^{-1}T, T] \times \mathbb{R}^n \to \mathbb{R}^n$ $(k \geq 1)$ of equation (10) by $y_k(t, a) =$

$$\begin{cases} a & if\ t \in [-k^{-1}T, 0], \\ e^{-t}\left[a + \int_0^t e^s h(s, a)\, ds\right] & if\ t \in (0, k^{-1}T], \\ e^{-t}\left[a + \int_0^t e^s h(s, y_k(s - k^{-1}T, a))\, ds\right] & if\ t \in ((j-1)k^{-1}T, jk^{-1}T], \end{cases} \tag{11}$$

where $j = 2, \ldots, k$. By construction, each y_k is continuous on $[-k^{-1}T, T] \times \mathbb{R}^n$, and satisfies the identity

$$y_k(t, a) = e^{-t}\left[a + \int_0^t e^s h\left(s, y_k(s - k^{-1}T, a)\right)\, ds\right], \quad t \in [0, T]. \tag{12}$$

Furthermore, for $k = 1, 2, \ldots$,

$$y_k'(t, a) = y_k(t, a) + h\left(t, y_k(t - k^{-1}T, a)\right), \quad t \in [-k^{-1}T, T]. \tag{13}$$

3.3. The modified problem (8) has at least one solution.

Let $(y_k)_{k \in \mathbb{N}}$ be the sequence of functions defined by Stampacchia's algorithm (11). We first show that for each $k \geq 1$, there exists at least one $a_k \in \mathbb{R}^n$ such that

$$y_k(T, a_k) = y_k(0, a_k). \tag{14}$$

Using formula (12), condition (14) can be written

$$a_k = e^{-T}a_k + \int_0^T e^{-(T-s)} h\left(s, y_k(s - k^{-1}T, a_k)\right)\, ds.$$

and hence

$$a_k = (1 - e^{-T})^{-1} \int_0^T e^{-(T-s)} h\left(s, y_k(s - k^{-1}T, a_k)\right)\, ds. \tag{15}$$

The mapping $F_k : \mathbb{R}^n \to \mathbb{R}^n$ defined by the right-hand member of (15) is continuous and bounded by a constant $R > 0$ independent of k. So, the Brouwer fixed point theorem implies the existence of at least one fixed point $a_k^* \in \overline{B}(R)$.

Next, we prove that the family $\{y_k(\cdot, a_k^*) : k = 1, 2, \ldots\}$ is relatively compact in $C([0, T], \mathbb{R}^n)$, or equivalently, using the Ascoli-Arzelá theorem, is bounded and equicontinuous. The boundedness is a direct consequence of Stampacchia's construction of y_k and the boundedness of h. For the equicontinuity, it follows from (13) and the boundedness of h that there exists $S > 0$

such that $\|y_k'(t, a_k^*)\| \le S$ for all $k = 1, 2, \ldots$. Thus, going if necessary to a subsequence, we can assume that $(y_k(\cdot, a_k^*))_{k\in\mathbb{N}}$ converges uniformly on $[0, T]$ to some $y^* \in C([0, T], \mathbb{R}^n)$. In particular, $a_k^* = y_k(0, a_k^*) \to y^*(0)$ if $k \to \infty$. From relation (14), we deduce that

$$y^*(T) = y^*(0),$$

i.e. that $y*$ satisfies the T-periodic boundary condition. On the other hand, using identity (12) and Lebesgue dominated convergence theorem, we obtain

$$y^*(t) = e^{-t}\left[y^*(0) + \int_0^t e^s h(s, y^*(s)), ds\right], \quad t \in [0, T],$$

which means that y^* is of class C^1 and is solution of the differential equation in (8).

3.4. The original problem (7) has at least one solution x such that $x(t) \in C$ for all $t \in [0, T]$.

By the previous claim we know that the modified problem (8) has at least one solution $x(t)$. Let us show that $x(t) \in C$ for all $t \in [0, T]$. The set I_+ of $t \in [0, T]$ such that $x(t) \in \mathbb{R}^n \setminus C$, that is such that $\|x(t) - p_C(x(t)\| > 0$, is open in $[0, T]$, and if $t \in I_+$, we have, using Lemma 2.1

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|x(t) - p_C(x(t))\|^2\right] = \frac{d}{dt}\delta_C(x(t)) = \langle \nabla\delta_C(x(t)), x'(t)\rangle \\
&\quad = \langle x(t) - p_C(x(t)), -[x(t) - p_C(x(t))] + f(t, p_C(x(t)))\rangle \\
&\quad = -\|x(t) - p_C(x(t))\|^2 + \langle x(t) - p_C(x(t)), f(t, p_C(x(t)))\rangle \\
&\quad \le \|x(t) - p_C(x(t))\|^2 < 0,
\end{aligned} \tag{16}$$

because $\dfrac{x - p_C(x)}{\|x - p_C(x)\|}$ is an outer normal to ∂C at $p(x)$ when $x \in \mathbb{R}^n \setminus C$, so that, by assumption (4),

$$\begin{aligned}
&\langle x(t) - p_C(x(t)), f(t, p_C(x(t))\rangle \\
&\quad = \|x(t) - p_C(x(t))\| \left\langle \frac{x(t) - p_C(x(t))}{\|x(t) - p_C(x(t))\|}, f(t, p_C(x(t)))\right\rangle \le 0.
\end{aligned}$$

Let $\tau \in [0, T]$ be such that

$$\frac{1}{2}\|x(\tau) - p_C(x(\tau))\|^2 = \max_{t\in[0,T]}\left[\frac{1}{2}\|x(t) - p_C(x(t))\|^2\right].$$

If $\tau \in I_+ \setminus \{0, T\}$, then

$$\frac{d}{dt}\left[\frac{1}{2}\|x(\tau) - p_C(x(\tau))\|^2\right] = 0,$$

a contradiction to (16). If $\tau = 0$, then, by the periodic boundary condition, the maximum is also reached in $\tau = T$, so that

$$\frac{d}{dt}\left[\frac{1}{2}\|x(T) - p_C(x(T))\|^2\right] \ge 0,$$

a contradiction to (16). Thus $\tau \in [0, T] \setminus I_+$,

$$0 \le \|x(t) - p_C(x(t))\|^2 \le \|x(\tau) - p_C(x(\tau))\|^2 = 0,$$

so that $x(t) \in C$ for all $t \in [0,T]$ and is a solution of the problem (7).

Remark 3.1. Condition (4) is the key to prove that any *periodic* solution of the modified problem (8) belongs in fact to C and then it is also a solution of the original problem (7).

We point out that (4) is equivalent to

$$f(t,x) \in T_C(x) \ \text{ for all } \ (t,x) \in [0,T] \times \partial C,$$

so it is justified to call it a "tangency condition".

4. The equivalence between Theorem 1.1 and Brouwer FPT

Since Theorem 1.1 is a consequence of the Brouwer fixed point theorem, we only have to prove the reciprocal implication to finish the proof of Theorem 1.2. Let C be a nonempty, closed, bounded and convex set and $g : C \to C$ a continuous function. Consider the autonomous differential equation

$$x'(t) = f(x(t)), \quad \text{where } f(x) = g(x) - x. \tag{17}$$

Let $\nu(x)$ be a normal outer field on ∂C. Then

$$\langle \nu(x), f(x)\rangle = \langle \nu(x), g(x) - x\rangle \leq 0 \quad \text{ for all } x \in \partial C,$$

since $g(\partial C) \subset C$ and so condition (4) is satisfied. Now, Theorem 1.1 provides a $T-$periodic solution contained in C for each $T > 0$ (note that as f is autonomous then it is $T-$periodic for each $T > 0$).

Therefore, for each $k \in \mathbb{Z}^+$ let x_k be a $\frac{1}{2^k}-$periodic solution of (17) contained in C. Clearly, x_k satisfies

$$x_k(t) = x_k(0) + \int_0^t f(x_k(s))ds, \tag{18}$$

and thus from the Arzelà–Ascoli theorem the sequence (x_k) is relatively compact, so it has a subsequence converging uniformly to x^* in C. Passing to the limit in (18) we obtain

$$x_*(t) = x_*(0) + \int_0^t f(x_*(s))ds.$$

Therefore, x^* is also a solution of (17) and moreover it is $\frac{1}{2^k}-$periodic for each $k \in \mathbb{Z}^+$ (since x_m is $\frac{1}{2^k}-$periodic for each $m \geq k$). As a periodic continuous function without minimal period is constant, we get that x^* is a constant solution of (17) and then

$$0 = f(x^*) = g(x^*) - x^*,$$

so x^* is a fixed point of g.

5. Final remarks and zeros of mappings in convex sets

In the original formulation of Gustafson and Schmitt [7], the condition upon C is stronger (C is a bounded, closed, convex neighborhood of the origin) and the inequality in assumption (4) is supposed to be strict, but to hold only for *some* outer normal at x. The strict character of the inequality is easy to remove, but we do not know how to deduce Theorem 1.1 from the Brouwer fixed point theorem when condition (4) holds for one outer normal field only.

The same idea as in the proof of the second part of Theorem 1.2 given in Section 4 allows to strengthen the conclusion of Theorem 1.1 in case f is autonomous.

Corollary 5.1. *Suppose that $C \subset \mathbb{R}^n$ is a nonempty, closed, bounded and convex set, $f : C \to \mathbb{R}^n$ is a continuous function and that, for each outer normal field $\nu : \partial C \to S^{n-1}$, the condition*

$$\langle \nu(x), f(x) \rangle \leq 0 \ \text{ for all } \ x \in \partial C \tag{19}$$

holds. Then, the differential equation $x'(t) = f(x(t))$ has a constant solution in C.

Clearly, Corollary 5.1 can be reformulated as a sufficient condition for the existence of a zero of f. There is a large literature on analogous tangency conditions like (19), in particular, in the case of a compact convex set C, which can be traced back at least to Browder [3] and Hartman and Stampacchia [8].

In the paper [18], simple proofs of the Hadamard and Poincaré-Miranda theorems for the existence of zeros of continuous mappings in a closed ball or a n-dimensional interval of $\mathbb{R}^n$ had been given, based upon the application of the Brouwer fixed point theorem on a closed ball of $\mathbb{R}^n$ to suitably associated fixed point problems. Theorem 1.1 can be seen as an adaptation of similar ideas to $T-$periodic solutions of ordinary differential equations and it suggests a similar proof of the following slight improvement of a common generalization of the Hadamard and Poincaré-Miranda theorems proved by Fonda and Gidoni in [6, Theorem 3], using the Brouwer degree, when C has a nonempty interior.

Theorem 5.1. *If $C \subset \mathbb{R}^n$ is a nonempty, closed, bounded and convex set, then any continuous mapping $f : C \to \mathbb{R}^n$ such that*

$$f(x) \notin N_x \setminus \{0\} \quad \text{ for all } x \in \partial C \tag{20}$$

has a zero in C.

Proof. Let us define the continuous mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(x) = p_C(x) + f(p_C(x)).$$

Then, for each $x \in \mathbb{R}^n$,

$$\|g(x)\| \leq \operatorname{diam} C + \max_{y \in C} \|f(y)\| := R.$$

So $g : \overline{B}_R \to \overline{B}_R$ has a fixed point $x^* \in \overline{B}_R$, that is, $f(p_C(x^*)) = x^* - p_C(x^*)$. If $x^* \in \mathbb{R}^n \setminus C$ then $p_C(x^*) \in \partial C$ and $x^* - p_C(x^*) \in N_{p_C(x^*)} \setminus \{0\}$, a contradiction. Hence, $x^* \in C$ and $f(x^*) = 0$. □

Remark 5.1. The referee pointed out that Theorem 5.1 can be obtained as a special case of Theorem (2.5) in the survey by Kryszewski [11]. However a direct proof in this simpler framework might still be of value.

Thus, Theorem 1.1 can be seen like a "nonautonomous" version of Theorem 5.1. However, while in the autonomous case it is enough to ask the "avoiding normal cone" condition (20) to ensure the existence of a constant solution in the nonautonomous setting we need the stronger "tangency condition" (4) in order to get the existence of a solution for the periodic boundary value problem (7).

By using the Brouwer degree, it is easy to prove the following variant of Corollary 5.1.

Corollary 5.2. *If $C \subset \mathbb{R}^n$ is a closed, bounded and convex neighborhood of* 0*, then any continuous function $f : C \to \mathbb{R}^n$ such that the condition (19) holds for some outer normal field $\nu : \partial C \to S^{n-1}$ has a zero in C.*

Proof. Firstly observe that for all $x \in \partial C$ it is satisfied that

$$\langle \nu(x), x\rangle \geq r := \text{dist}(0, \partial C) > 0.$$

Indeed, since $r\nu(x) \in C$ from the definition of an outer normal field we have

$$0 \geq \langle \nu(x), r\nu(x) - x\rangle = r - \langle \nu(x), x\rangle \quad \text{ for all } x \in \partial C,$$

from which the result holds.

Consider now the linear homotopy

$$F(x, \lambda) = (1 - \lambda)x - \lambda f(x),\ \lambda \in [0, 1),$$

and compute, taking into account condition (19), for $x \in \partial C$ and $\lambda \in [0, 1)$

$$\langle \nu(x), F(x, \lambda)\rangle = (1 - \lambda)\langle \nu(x), x\rangle - \lambda\langle \nu(x), f(x)\rangle \geq (1 - \lambda)r > 0.$$

Then, either f has a zero $x \in \partial C$ or $F(x, \lambda) \neq 0$ for all $x \in \partial C$ and $\lambda \in [0, 1]$. In this last case, by using the properties of the Brouwer degree d_B in the open, bounded set $\Omega = \text{int(C)}$ containing zero (note that $\partial\Omega = \partial C$ since C is convex), we obtain

$$d_B(f, \Omega) = d_B(F(x, 1), \Omega) = d_B(F(x, 0), \Omega) = d_B(Id, \Omega) = 1,$$

which implies the existence of a zero of f in C. □

Remark 5.2. Note that in Corollary 5.2 the inequality (19) is only requested for one outer normal field but the set C should have nonempty interior. We do not know if this version can be deduced from the Brouwer fixed point theorem.

For a more detailed study on the connections between the Brouwer fixed point theorem and the existence of zeros for certain functions the reader is referred to [13, 19, 20].

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