AN ABSTRACT AVERAGING METHOD WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Abstract. We present a general formulation of the averaging method in the setting of a semilinear equation $Lx = \varepsilon N(x, \varepsilon)$, being L a linear Fredholm mapping of index zero. Our general approach provides new results even in the classical periodic framework. Among the applications we obtained there are: a partial answer to an open problem related to the Liebau phenomenon, the multiplicity of periodic solutions for a planar system with delay and the existence of solution for a nonlocal chemical reactor.

1. Introduction

The *averaging method* is a fruitful technique in perturbation theory that goes back to the classical works on celestial mechanics of Clairaut, Lagrange and Laplace in the XVIIIthcentury. Through the XIXth century it progressed in the skillful hands of renowned mathematicians, such as Jacobi and Poincaré, until its rigorous formalization by Fatou in 1928, and its substantial development and application to nonlinear mechanics in the 1930's by the Kiev school of mathematics leaded by Krylov, Bogoliubov, and Mitropolsky (see [23]).

As pointed out in [8, Chapter 4], there are many different averaging theorems, whose main goal is to obtain information about a nonlinear system of differential equations with a small parameter ε

$$x' = \varepsilon f(t, x, \varepsilon), \quad \text{where } f \text{ is } T\text{-periodic in } t, \tag{1}$$

from the *averaged* autonomous system

$$x' = \varepsilon F(x), \quad \text{where} \quad F(x) = \tfrac{1}{T}\int_0^T f(t, x, 0)dt. \tag{2}$$

The information transferred from problem (2) to problem (1) can be mainly of two types : asymptotics estimates for the solutions of initial value problems on a large or infinite interval (see [23]), or the existence of T-periodic solutions for (1) associated to a critical point of (2) (see [10, Chapter V, Theorem 3.2] or [27, Page 168]).

In this paper we focus on an abstract version of the second approach motivated by the extension of a result given in [29] for a frictionless equation related to the Liebau phenomenon to an equation containing a friction term. To reach this goal we develop an averaging theorem for the existence of periodic solutions not covered by the previous literature which in fact is a special case of a more general result

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formulated in the framework of small nonlinear perturbations of *Fredholm linear mappings of index zero.* Roughly speaking, while searching for T-periodic solutions of equation (1) is equivalent to solve a semilinear equation of the type

$$Lx = \varepsilon N(x, \varepsilon),$$

in some infinite dimensional space, with L a Fredholm operator of index zero, we can take advantage of the fact that the critical points of (2) are solutions of a finite dimensional system defined in some subset of $\mathrm{Ker}(L)$. Besides this our abstract formulation extends the applicability of the averaging method from periodic solutions to other kinds of boundary conditions.

The paper is organized as follows: in Section 2 we present our main averaging theorem in an abstract setting. Section 3 is devoted to develop a periodic version of our general averaging. Firstly, for ordinary differential equations, a result that is even new up to our knowledge, and we present two applications: one to the Liebau phenomenon and another one to a pendulum with oscillating support. Secondly, we also deal with the periodic averaging for functional differential equations and we derive the existence of multiple periodic solutions for a planar system with delay. Finally, in Section 4 we develop an averaging method for a problem with nonlocal and nonlinear boundary conditions. A special case with nonlocal linear boundary conditions, which extends the Neumann ones, is analyzed in more detail and we obtain as a consequence an application to a tubular chemical reactor.

2. The averaging method for abstract semilinear equations

2.1. Some algebraic preliminaries. For the convenience of the reader we collect here some basic definitions and properties of Fredholm mappings. A more detailed account can be found in the monographs [17, 18].

Let X and Z be real Banach spaces, $\Omega \subset X$ an open set, $\varepsilon_1 > 0$, $L : X \to Z$ and $N : \Omega \times (-\varepsilon_1, \varepsilon_1) \to Z$ such that:

- (H0) L is linear, continuous and $\mathrm{Im}(L)$ is closed in Z.
- (H1) $\mathrm{Ker}(L)$ and $\mathrm{Coker}(L) = Z/\mathrm{Im}(L)$ have the same finite dimension.
- (H2) N is continuous in $\Omega \times (-\varepsilon_1, \varepsilon_1)$ and, for each $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $N(\cdot, \varepsilon)$ is Fréchet differentiable in Ω, and $\frac{\partial N}{\partial x}$ is continuous at $(x_0, 0)$.

A mapping L satisfying (H0) and (H1) is called a *Fredholm mapping of index zero.* For such operators there exist continuous projectors (i.e., linear bounded and idempotent operators) $P : X \to X$ and $Q : Z \to Z$ such that the sequence

$$X \xrightarrow{P} X \xrightarrow{L} Z \xrightarrow{Q} Z$$

is exact, that is, $\mathrm{Im}(P) = \mathrm{Ker}(L)$ and $\mathrm{Im}(L) = \mathrm{Ker}(Q)$. Then,

$$X = \mathrm{Ker}(L) \oplus \mathrm{Ker}(P) \quad \text{and} \quad Z = \mathrm{Im}(L) \oplus \mathrm{Im}(Q),$$

and the restriction

$$L_P : \mathrm{Ker}(P) \to \mathrm{Im}(L)$$

is an isomorphism. So its algebraic inverse $K_P : \mathrm{Im}(L) \to \mathrm{Ker}(P)$ is well defined.

2.2. **The averaging method.** Let us consider the problem

$$Lx \quad = \quad \varepsilon N(x,\varepsilon), \quad x \in \Omega, \quad \varepsilon \in (-\varepsilon_1, \varepsilon_1), \tag{3}$$

where L and N are as in the previous section. Let us define the averaged function

$$F : \mathrm{Ker}(L) \cap \Omega \to \mathrm{Im}(Q), \quad x \mapsto F(x) := Q(N(x,0)).$$

Notice that $\mathrm{Ker}(L)$ has finite dimension, so the *averaged equation*

$$F(x) = 0, \quad x \in \mathrm{Ker}(L) \cap \Omega,$$

is finite dimensional. Moreover, $F \in C^1$ and, for all $x \in \mathrm{Ker}(L) \cap \Omega$,

$$F'(x) = Q \circ \frac{\partial N}{\partial x}(x,0).$$

The following is our main result, an existence theorem for the abstract equation (3) based on the averaging method.

Theorem 1. *Under the assumptions (H0), (H1) and (H2), if there exists $x_0 \in$ Ker(L) such that*

$$F(x_0) = 0 \quad \textit{and} \quad F'(x_0) \textit{ is an isomorphism,} \tag{4}$$

then there exists $\varepsilon_0 \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}$ such that, for $0 < |\varepsilon| < \varepsilon_0$, the problem (3) has a unique solution $x(t,\varepsilon)$ in a sufficiently small neighborhood of x_0. Moreover

$$\lim_{\varepsilon \to 0} \|x(t,\varepsilon) - x_0\|_X = 0.$$

Proof. Let us define the nonlinear operator $G : \Omega \times (-\varepsilon_1, \varepsilon_1) \to Z$ by

$$G(x,\varepsilon) = Lx - (1-\varepsilon)Q(N(x,\varepsilon)) - \varepsilon N(x,\varepsilon).$$

If

$$G(x,\varepsilon) = 0, \tag{5}$$

then

$$0 = Q(G(x,\varepsilon)) = -Q(N(x,\varepsilon)),$$

and hence

$$Lx = \varepsilon N(x,\varepsilon),$$

so that x is a solution of (3). Thus solutions of (5) give solutions of (3).

For $\varepsilon = 0$, problem (5) becomes

$$Lx = Q(N(x,0))$$

and then $x \in \mathrm{Ker}(L)$ and $F(x) := Q(N(x,0)) = 0$ (remember that $Z = \mathrm{Im}(L) \oplus \mathrm{Im}(Q)$). Conversely, all zeros of F are solutions of $G(x,0) = 0$.

Because G has the same regularity as N, to apply the implicit function theorem in Banach spaces [30, Theorem 4.B] to equation (5) near $(x_0, 0)$, we must study the invertibility of the partial derivative of G with respect to x at $(x_0, 0)$. For any $v \in \Omega$, we have

$$\frac{\partial G}{\partial x}(x,\varepsilon)v = Lv - (1-\varepsilon)Q\left(\frac{\partial N}{\partial x}(x,\varepsilon)v\right) - \varepsilon\frac{\partial N}{\partial x}(x,\varepsilon)v.$$

and hence,

$$\frac{\partial G}{\partial x}(x_0,0)v = Lv - Q\left(\frac{\partial N}{\partial x}(x_0,0)v\right)$$

For any $v \in X$, let us write $v = \widehat{v} + \widetilde{v}$, where $\widehat{v} \in \mathrm{Ker}(L)$ and $\widetilde{v} \in \mathrm{Ker}(P)$. Analogously, for any $h \in Z$, let us write $h = \widehat{h} + \widetilde{h}$, where $\widehat{h} \in \mathrm{Im}(Q)$ and $\widetilde{h} \in \mathrm{Im}(L)$. For $h \in Z$, the equation

$$\frac{\partial G}{\partial x}(x_0, 0)v = h, \quad v \in X, \tag{6}$$

is equivalent to

$$Lv - Q\left(\frac{\partial N}{\partial x}(x_0, 0)v\right) = h, \quad v \in X,$$

that is,

$$L\widehat{v} + L\widetilde{v} - Q\left(\frac{\partial N}{\partial x}(x_0, 0)v\right) = \widehat{h} + \widetilde{h}.$$

Clearly, $L\widehat{v} = 0$, and since $Z = \mathrm{Im}(L) \oplus \mathrm{Im}(Q)$ we obtain the equivalent Lyapunov-Schmidt system

$$-Q\left(\frac{\partial N}{\partial x}(x_0, 0)v\right) = \widehat{h}, \tag{7}$$

$$L\widetilde{v} = \widetilde{h}. \tag{8}$$

As noted in Section 2.1 we know that L is an isomorphism from $\mathrm{Ker}(P)$ to $\mathrm{Im}(L)$, and hence the bifurcation equation (8) has a unique solution $\widetilde{v} = K_P\widetilde{h}$ in $\mathrm{Ker}(P)$, which, introduced in the equation (7), gives

$$Q\left(\frac{\partial N}{\partial x}(x_0, 0)\widehat{v}\right) = -\widehat{h} - Q\left(\frac{\partial N}{\partial x}(x_0, 0)\widetilde{v}\right),$$

i.e.

$$F'(x_0)\widehat{v} = -\widehat{h} - Q\left(\frac{\partial N}{\partial x}(x_0, 0)\widetilde{v}\right).$$

Therefore, if x_0 is such that $F'(x_0)$ is an isomorphism, the problem (6) has a unique solution $v \in X$ for each $h \in Z$, and the conditions of the implicit function theorem in Banach spaces are satisfied at $(x_0, 0)$. The conclusion follows from this theorem. □

Remark 2. Recently some papers have dealt with the existence of periodic solutions for problems with a small parameter by means of degree theory, see for instance [2, 14]. This type of results combining the averaging method with degree theory can be traced back at least to Cronin's monograph [6] (see also [19] for a historical account) and an abstract formulation in the setting of small perturbations of linear index zero Fredholm mappings can be found in [7, Theorem IV.II]: typically, instead of condition (4) it is asked that

$$d_B[F, D, 0] \neq 0, \tag{9}$$

where D is a bounded open subset of $\mathbb{R}^n$ such that $0 \notin F(\partial D)$ and d_B is the Brouwer degree. This condition implies in particular the existence of $x_0 \in D$ such that $F(x_0) = 0$. On the other hand, (4) implies that x_0 is an isolated zero of F and that, for D a sufficiently small open neighborhood of x_0,

$$d_B[F, D, 0] = \mathrm{sign}\,(\det\,(F'(x_0))) \neq 0.$$

Of course, what is lost with the degree condition (9) is the local uniqueness of the periodic solution $x(t,\varepsilon)$ and hence the fact that $x(t,\varepsilon) \to x_0$ when $\varepsilon \to 0$. Such information can be critical in some applications, as for instance in [28, 29] where the averaging method was combined with the third order approximation to obtain stability information about the periodic solutions. It is worth noting that the idea of transforming a problem of finding periodic solutions of $x' = \varepsilon f(t,x,\varepsilon)$ in terms of semilinear equations of the type $Lx = \varepsilon N(x,\varepsilon)$ like in Theorem 1 has a long history, and is for example emphasized in Chapter IX of [10], in [16], and, more recently in Section 4 of the interesting paper [2], where coincidence degree techniques are used under more general assumptions which exclude local uniqueness conclusions and an iteration method for the obtained T-periodic solution.

Remark 3. The proof of the implicit function theorem for equation $G(x,\varepsilon) = 0$ when $G(x_0, 0) = 0$ [30, Theorem 4.B] consists in defining the operator $H : X \times (-\varepsilon_1, \varepsilon_1) \to X$ by

$$H(x,\varepsilon) = x - \left[\frac{\partial G}{\partial x}(x_0, 0)\right]^{-1} G(x,\varepsilon)$$

(such that the fixed points of $H(\cdot,\varepsilon)$ are the zeros of $G(\cdot,\varepsilon)$) and proving that $H(\cdot,\varepsilon)$ is a contraction mapping of the closed ball $B_{x_0}(\delta_0)$ into itself for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, when δ_0 and ε_0 are sufficiently small. The Banach fixed point theorem implies then the existence of a unique fixed point $x(\varepsilon)$ of $H(\cdot,\varepsilon)$ in $B_{x_0}(\delta_0)$. Furthermore, $\lim_{\varepsilon\to 0} x(\varepsilon) = x_0$ and $x(\varepsilon) = \lim_{k\to\infty} x_k(\varepsilon)$, where $x_0(\varepsilon) = x_0$ and, for each $k = 0, 1, \ldots$,

$$x_{k+1}(\varepsilon) = x_k(\varepsilon) - \left[\frac{\partial G}{\partial x}(x_0, 0)\right]^{-1} G(x_k(\varepsilon), \varepsilon).$$

This iteration can easily be written in terms of the original mappings L and N.

3. Applications to periodic solutions

Throughout this section we are going to use the following notation: for a given continuous function $h : [0,T] \to \mathbb{R}$ we denote by $\bar{h}$ its mean value, that is,

$$\bar{h} = \frac{1}{T}\int_0^T h(s)ds.$$

3.1. **Periodic averaging for ordinary differential equations.** Let $n \geq 1$ be an integer, $T > 0$, $I \subset \mathbb{R}$ an open interval and

$$\begin{aligned} f : [0,T] \times I \times \mathbb{R} \times \ldots \times \mathbb{R} \times (-\varepsilon_1, \varepsilon_1) &\to \mathbb{R}, \\ (t, u_0, u_1, \ldots, u_{n-1}, \varepsilon) &\mapsto f(t, u_0, u_1, \ldots, u_{n-1}, \varepsilon) \end{aligned}$$

be continuous and such that, for each $k = 0, \ldots, n-1$, $\dfrac{\partial f}{\partial u_k}$ exists and is continuous.

Given $a_1, a_2, \ldots, a_{n-1}$ in $\mathbb{R}$, let us consider the problem

$$\begin{aligned} x^{(n)} + \sum_{j=1}^{n-1} a_{n-j} x^{(n-j)} &= \varepsilon f(t, x, x', \ldots, x^{(n-1)}, \varepsilon), \\ x^{(j)}(0) &= x^{(j)}(T) \ \ (j = 0, \ldots, n-1), \end{aligned} \tag{10}$$

and define the periodic averaged function

$$F : I \to \mathbb{R}, \quad c \mapsto F(c) := \frac{1}{T}\int_0^T f(s, c, 0, \dots, 0, 0)\, ds. \tag{11}$$

Then $F \in C^1(I, \mathbb{R})$ and, for all $c \in I$

$$F'(c) = \frac{1}{T}\int_0^T \frac{\partial f}{\partial u_0}(s, c, 0, \dots, 0, 0)\, ds.$$

The following result generalizes the classical periodic averaging method, [9], even for second order equations.

Theorem 4. *Assume that the linear problem*

$$x^{(n)} + \sum_{j=1}^{n-1} a_{n-j} x^{(n-j)} = 0, \;\; x^{(j)}(0) = x^{(j)}(T),\; j = 0, \dots, n-1,$$

has only constant solutions.

Then, for each $c_0 \in I$ such that

$$F(c_0) = 0 \quad \textit{and} \quad F'(c_0) \neq 0,$$

where F is given by (11), there exists $\varepsilon_0 \in [-\varepsilon_1, \varepsilon_1] \setminus \{0\}$ such that, for $0 < |\varepsilon| < \varepsilon_0$, the problem (10) has a unique solution $x(t, \varepsilon)$ such that $\lim_{\varepsilon \to 0} x(t, \varepsilon) = c_0$ uniformly in $t \in [0, T]$.

Proof. Let $X = C^n_T[0, T]$ be the subspace of elements of $C^n[0, T]$ such that $x^{(j)}(0) = x^{(j)}(T)$ $(j = 0, \dots, n-1)$, $Z = C[0, T]$,

$$Lx = x^{(n)} + \sum_{j=1}^{n-1} a_{n-j} x^{(n-j)},$$

and

$$N(x, \varepsilon) = f(t, x, x', \dots, x^{(n-1)}, \varepsilon).$$

L is a Fredholm mapping of index zero as a special case of [7, Proposition IX.2, p. 171-172] but we explicit the details for the reader's convenience: by Proposition IX.1 in [7] the assumption upon L implies that, with $\omega := \frac{2\pi}{T}$,

$$(im\omega)^n + \sum_{j=1}^{n-1} a_{n-j}(im\omega)^{n-j} \neq 0 \;\text{ for any integer }\; m \neq 0. \tag{12}$$

The adjoint operator L^* of L, such that $\langle Lu, v\rangle = \langle u, L^* v\rangle$, where $\langle x, y\rangle := \int_0^T x(t)y(t)\, dt$, is defined by

$$L^* u = (-1)^n u^{(n)} + \sum_{j=1}^{n-1} (-1)^{n-j} a_{n-j} u^{(n-j)}.$$

Clearly,

$$(-1)^n (im\omega)^n + \sum_{j=1}^{n-1} (-1)^{n-j} a_{n-j}(im\omega)^{n-j} \neq 0 \;\text{ for any integer }\; m \neq 0$$

if and only if condition (12) holds, as immediately seen by replacing m by $-m$ in (12). Consequently, defining the projectors $Px = \dfrac{1}{T}\displaystyle\int_0^T x(s)ds$ for $x \in X$ and $Qz = \dfrac{1}{T}\displaystyle\int_0^T z(s)ds$ for $z \in Z$, we have that

$$\mathrm{Ker}(L^*) = \mathrm{Ker}(L) = \mathrm{Im}(P),$$

and using the Fredholm alternative for the forced linear problem (see e.g. Proposition IX.1 in [7]), we obtain

$$\mathrm{Im}(L) = \mathrm{Ker}(Q).$$

Hence the sequence

$$C^n_T[0,T] \xrightarrow{P} C^n_T[0,T] \xrightarrow{L} C[0,T] \xrightarrow{Q} C[0,T],$$

is exact and all the assumptions of Theorem 1 are satisfied. □

Remark 5. An alternative to the above proof, which is less elementary but does not request the Fredholm alternative for linear periodic problems, is based upon the following result of the theory of Fredholm operators (see e.g. [1, Corollary 28Q] or [30, Proposition 8.14 (3)]) : *If X and Z are Banach spaces, $A : X \to Z$ is a Fredholm mapping of index zero and $B : X \to Z$ is a compact linear operator, then $A + B$ is a Fredholm mapping of index zero.* With the notations of Theorem 4, define

$$A : C^n_T[0,T] \to C[0,T], \quad x \mapsto x^{(n)},$$

and

$$B : C^n_T[0,T] \to C[0,T], \quad x \mapsto \sum_{j=1}^{n-1} a_{n-j}x^{(n-j)}.$$

Using Ascoli-Arzelá's theorem, it is easy to show that B is compact. On the other hand, it is immediately checked that $\mathrm{Ker}(A) = \mathrm{Im}(P)$ and $\mathrm{Im}(A) = \mathrm{Ker}(Q)$, because if $h \in C[0,T]$ is such that $\displaystyle\int_0^T h(t)\,dt = 0$, the solutions of the periodic problem

$$x^{(n)}(t) = h(t), \quad x^{(j)}(0) = x^{(j)}(T),\ j = 0, \dots, n-1,$$

are given by $x(t) = Px + (K^n h)(t)$, where K is the operator which associates to any T-periodic function with mean value zero its T-periodic primitive with mean value zero, namely

$$(Kh)(t) = \int_0^t h(s)\,ds - \frac{1}{T}\int_0^T \left[\int_0^\tau h(s)\,ds\right] d\tau.$$

Hence, $L = A + B$ is a Fredholm mapping of index zero with $\dim \mathrm{Ker}(L) = \mathrm{codim}\ \mathrm{Im}(L) = 1$, and as, trivially, $\mathrm{Im}(L) \subset \mathrm{Ker}(Q)$ and $\mathrm{codim}\ \mathrm{Ker}(Q) = 1$, we have $\mathrm{Im}(L) = \mathrm{Ker}(Q)$.

Remark 6. For the n-th order equation with a small parameter

$$x^{(n)} = \varepsilon f(t, x, \varepsilon),$$

and periodic boundary conditions

$$x^{(j)}(0) = x^{(j)}(T), \quad j = 0, \dots, n-1,$$

an existence theorem based on the averaging method can be found in [9, Pages 105–106]. Moreover [22, Corollary 2.5] provides a convergent method of successive approximations for the solution $x(t,\varepsilon)$ of the first order equation

$$x' = \varepsilon f(t,x,\varepsilon),$$

subject to $x(0) = x(T)$. The case of first order systems can be found for instance in [10, 17]. However, as far as we know, Theorem 4 is new in the related literature, giving existence, asymptotic behavior and an iteration method for the solution from a simpler proof and in a more general setting. An anonymous referee of this paper pointed out that the existence and asymptotic behavior part of the result could also be obtained from the more general version of the averaging method recently given by Llibre and Novaes in [15] for problems of the form

$$x' = F_0(t,x) + \varepsilon F_1(t,x) + O(\varepsilon^2),$$

by reducing the scalar n^{th} order differential equation in problem (10) to a first order system in $\mathbb{R}^n$ in the standard way $x = x_1, x' = x_2, \ldots, x^{(n-1)} = x_n$, and calling $F_0(t, x_1, \ldots, x_{n-1}, x_n)$ the part of the right-hand member independent of ε and depending upon the a_j. Of course, the main achievement of the results of [15] lies more in its capacity in treating perturbations of order ε of nonlinear differential systems.

3.1.1. *Application to the Liebau phenomenon.* The *valveless pumping effect* refers to a preferential direction of a fluid without the aid of valves due to an asymmetric periodic excitation. This effect is also called the *Liebau phenomenon*, since the German cardiologist G. Liebau conjectured that breathing could explain the unexpected effectiveness of blood circulation on the human beings, [13]. A model which exhibits the pumping effect, based on a simple configuration of one pipe and one tank, was developed in [20]. The reader is also referred to [26, Chapter 8] for a detailed account of the model: the problem reduces to searching positive T-periodic solutions for the singular second-order differential equation

$$u'' + a\,u' = \frac{1}{u}(e(t) - bu'^2) - c, \tag{13}$$

where $a \geq 0$, $b > 1$, $c > 0$ and e is continuous and T-periodic. Recently, some results on the existence and stability of periodic positive solutions for (13) were presented in [3, 4, 5, 12, 26, 29]. As pointed out in [3], $\bar{e} > 0$ is a necessary condition for the existence of a periodic positive solution of (13), and an open problem is to know if it is also sufficient. As a consequence of Theorem 4 we will give a partial answer to this question.

By means of the change of variables $u = x^\kappa$ with $\kappa = 1/(b+1)$, see [3], problem (13) can be rewritten as

$$x'' + a\,x' = \frac{e(t)}{\kappa}x^{1-2\kappa} - \frac{c}{\kappa}x^{1-\kappa}. \tag{14}$$

Notice that if we consider c as a small parameter, meaning that the section of the pipe is much smaller than the section of the tank, the equation (14) is of the form

$$x'' + ax' = r(t)x^\alpha - \varepsilon s(t)x^\beta, \tag{15}$$

where a, α, $\beta \in \mathbb{R}$ and r and s are T-periodic continuous functions. The following result extends [28, Theorem 2.1].

Theorem 7. *Assume that $a \in \mathbb{R}$, $\alpha \neq 1$, $\alpha \neq \beta$ and r and s are T-periodic continuous functions with $\overline{r} \cdot \overline{s} > 0$. Then equation (15) has a T-periodic solution $x(t, \varepsilon)$ provided that:*

(i) *Either $\dfrac{1-\alpha}{\beta-\alpha} > 0$ and $\varepsilon > 0$ is small enough,*

(ii) *or $\dfrac{1-\alpha}{\beta-\alpha} < 0$ and $\varepsilon > 0$ is large enough.*

Moreover, the following asymptotic behavior holds in both cases

$$\lim_{\varepsilon \to 0^+} \varepsilon^{\frac{1}{\beta-\alpha}} x(t, \varepsilon) = \left(\frac{\overline{r}}{\overline{s}}\right)^{\frac{1}{\beta-\alpha}} \quad \textit{uniformly in } t \in [0, T].$$

Proof. Setting $x = \mu^{\frac{1}{\alpha-1}} v$, equation (15) becomes, after simplification by $\mu^{\frac{1}{\alpha-1}}$,

$$v'' + av' = r(t)\mu v^\alpha - \varepsilon s(t) \mu^{\frac{\beta-1}{\alpha-1}} v^\beta,$$

and hence, choosing $\varepsilon = \mu^{\frac{\beta-\alpha}{1-\alpha}}$, we have

$$v'' + av' = \mu(r(t)v^\alpha - s(t)v^\beta). \tag{16}$$

Now, it is easy to show that equation (16) satisfies the conditions of Theorem 4. Indeed, for any $c > 0$ we have

$$F(c) = \overline{r}c^\alpha - \overline{s}c^\beta \quad \text{and} \quad F'(c) = \alpha \overline{r} c^{\alpha-1} - \beta \overline{s} c^{\beta-1},$$

and then for $c_0 = \left(\frac{\overline{r}}{\overline{s}}\right)^{\frac{1}{\beta-\alpha}} > 0$ it holds

$$F(c_0) = 0 \quad \text{and} \quad F'(c_0) = (\alpha - \beta) \frac{\overline{r}^{\frac{\beta-1}{\beta-\alpha}}}{\overline{s}^{\frac{\alpha-1}{\beta-\alpha}}} \neq 0.$$

□

Now, as an application of Theorem 7 to equation (13), through equation (14), we obtain the following extension of [29, Theorem 2.1 *(I)*] which partially answers the open problem stated above.

Corollary 8. *Let us assume $\overline{e} > 0$. Then there exists $c_0 > 0$ such that, for $0 < c < c_0$, problem (13) has a unique T-periodic solution $u(t, c)$ such that*

$$\lim_{c \to 0^+} c\, u(t, c) = \overline{e} \quad \textit{uniformly in } t \in [0, T].$$

3.1.2. *A pendulum with oscillating support.* Consider a pendulum attached to a moving cart as in [25, Section 5.1]. The angle $x(t)$ of the pendulum satisfies the following differential equation

$$x'' + ax' = \frac{1}{\ell}(-g \sin x - p''(t) \cos x), \tag{17}$$

where $a > 0$ is a viscous friction coefficient, $\ell > 0$ is the length of the pendulum, g is the gravitational acceleration and the function $p \in C^2(\mathbb{R})$, with p'' T-periodic, describes the motion of the cart. Let us set $\varepsilon = \frac{1}{\ell}$ in equation (17). Then, for $c \in \mathbb{R}$ we have

$$F(c) = -g \sin c - \overline{p''} \cos c \quad \text{and} \quad F'(c) = -g \cos c + \overline{p''} \sin c.$$

Observe that $F(c) = 0$ when $\tan c = -\frac{\overline{p''}}{g}$, and then $F'(c) \neq 0$. So, as consequence of Theorem 4, we obtain the following existence result.

Theorem 9. *There exists $\ell_0 > 0$ such that for $\ell > \ell_0$ equation (17) has a unique T-periodic solution $x(t, \ell)$ and*

$$\lim_{\ell \to +\infty} x(t, \ell) = c_0 \quad \textit{uniformly in } t \in [0, T],$$

where c_0 is the unique solution of the equation $\tan c_0 = -\frac{\overline{p''}}{g}$.

3.2. **Periodic averaging for functional differential equations.** Consider now the periodic boundary value problem for the functional differential equation

$$x'(t) = \varepsilon f(t, x_t, \varepsilon) \quad \text{for all } t \in [0, 1], \quad x(0) = x(1), \tag{18}$$

where for $x \in C([-r, 1], \mathbb{R}^n)$, $r \geq 0$, and $t \in [0, 1]$, we define

$$x_t(s) := x(s + t) \quad \text{for all } s \in [-r, 0],$$

and $f : [0, 1] \times C([-r, 0], \mathbb{R}^n) \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^n$, $(t, \varphi, \varepsilon) \mapsto f(t, \varphi, \varepsilon)$, is continuous, $\dfrac{\partial f}{\partial \varphi}$ exists and is continuous and $f(\cdot, \varphi, \varepsilon)$ is 1-periodic.

Let $X = \{x \in C^1(\mathbb{R}, \mathbb{R}^n) : x(t) = x(t + 1) \text{ for all } t \in \mathbb{R}\}$, $Z = C([0, 1], \mathbb{R}^n)$ and

$$L : X \to Z, \quad x \mapsto Lx = x'|_{[0,1]},$$

$$N : X \times (-\varepsilon_1, \varepsilon_1) \to Z, \quad (x, \varepsilon) \mapsto N(x, \varepsilon) = f(\cdot, x_\cdot, \varepsilon),$$

so that problem (18) is equivalent to the abstract equation $Lx = \varepsilon N(x, \varepsilon)$.

Now, it is easy to show that L is a Fredholm mapping of index zero, and, the averaged function for problem (18) is given by

$$F(c) := \int_0^1 f(s, c, 0)\, ds, \quad c \in \mathbb{R}^n. \tag{19}$$

Hence, as a direct consequence of Theorem 1 we obtain the following result.

Theorem 10. *For each $c_0 \in \mathbb{R}^n$ such that*

$$F(c_0) = 0 \quad \textit{and} \quad F'(c_0) \ \textit{is invertible},$$

where F is given by (19), there exists $\varepsilon_0 > 0$ such that, for $0 < |\varepsilon| < \varepsilon_0$, the problem (18) has a unique solution $x(t, \varepsilon)$ such that $\lim_{\varepsilon \to 0} x(t, \varepsilon) = c_0$ uniformly in $t \in [0, 1]$.

3.2.1. *A periodic planar delay-differential system.* As an example of application, we can consider the following planar delay-differential system (written in complex notations by letting $z = x_1 + ix_2$)

$$z'(t) = \varepsilon\left[z^m(t - r) - h(t)\right], \quad z(0) = z(1), \tag{20}$$

where $m \geq 1$ is an integer, $r \geq 0$ and $h : \mathbb{R} \to \mathbb{C}$ is continuous and 1-periodic. It is a special case of (18) with $n = 2$ and

$$f(t, \varphi, \varepsilon) = \varphi^m(-r) - h(t).$$

Consequently, with $c_0 \in \mathbb{C}$,

$$F(c_0) := \int_0^1 \left[c_0^m - h(t)\right] dt = c_0^m - \overline{h}, \tag{21}$$

and

$$\begin{aligned} \det F'(c_0) &= \det \begin{pmatrix} \Re(mc_0^{m-1}) & \Re(imc_0^{m-1}) \\ \Im(mc_0^{m-1}) & \Im(imc_0^{m-1}) \end{pmatrix} \\ &= m^2 \det \begin{pmatrix} \Re(c_0^{m-1}) & -\Im(c_0^{m-1}) \\ \Im(c_0^{m-1}) & \Re(c_0^{m-1}) \end{pmatrix} \\ &= m^2 \left((\Re(c_0^{m-1}))^2 + (\Im(c_0^{m-1}))^2 \right) \\ &= m^2 |c_0^{m-1}|^2 = m^2 |c_0|^{2m-2}. \end{aligned} \tag{22}$$

Corollary 11. *If $\overline{h} := |\overline{h}| \exp(i\theta) \neq 0$ then there exists $\varepsilon^* > 0$ such that for $0 < |\varepsilon| < \varepsilon^*$ the problem (20) has at least m solutions $z^{(j)}(t,\varepsilon)$ and*

$$z^{(j)}(t,\varepsilon) \to |\overline{h}|^{\frac{1}{m}} \exp\left(i \frac{\theta + 2j\pi}{m} \right) \text{ uniformly in } t \in [0,1], \quad j = 0,1,\ldots,m-1.$$

Proof. Since $\overline{h} := |\overline{h}| \exp(i\theta) \neq 0$, the m zeros of F in (21) are given by

$$c_0^{(j)} = |\overline{h}|^{\frac{1}{m}} \exp\left(i \frac{\theta + 2j\pi}{m} \right), \quad j = 0,1,\ldots,m-1,$$

and by (22) are such that

$$\det F'(c_0^{(j)}) = m^2 |c_0^{(j)}|^{2m-2} = m^2 |\overline{h}|^{\frac{2m-2}{m}} \neq 0, \quad j = 0,1,\ldots,m-1.$$

Applying Theorem 10 to each of the $c_0^{(j)}$ and calling ε^* the smallest of the ε_0 associated to each of them by Theorem 10, we obtain the result. □

4. Applications to nonlocal boundary value problems

4.1. Problems with nonlinear nonlocal boundary conditions.

Consider now the following problem for $a \in \mathbb{R}$, $a \neq 0$, with nonlocal and nonlinear boundary conditions

$$x'' + ax' = \varepsilon f(t, x, x', \varepsilon), \quad \Lambda[x] = \varepsilon C[x, \varepsilon], \quad \Theta[x] = \varepsilon D[x, \varepsilon], \tag{23}$$

where $f : [0,1] \times \mathbb{R}^2 \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$, $(t, u_0, u_1, \varepsilon) \mapsto f(t, u_0, u_1, \varepsilon)$ is continuous and such that, for $k = 0,1$, $\frac{\partial f}{\partial u_k}$ exists and is continuous, Λ, Θ are linear continuous functionals on $C^2([0,1])$ and C, D are continuous functionals on $C^2([0,1]) \times (-\varepsilon_1, \varepsilon_1)$ such that $C[\cdot, \varepsilon]$, $D[\cdot, \varepsilon]$ are of class C^1 on $C^2([0,1])$.

Let $X = C^2([0,1])$, $Z = C([0,1]) \times \mathbb{R}^2$,

$$L : X \to Z, \quad x \mapsto Lx := (x'' + ax', \Lambda[x], \Theta[x]),$$

and

$$N : X \times (-\varepsilon_1, \varepsilon_1) \to Z, \quad (x, \varepsilon) \mapsto N(x, \varepsilon) := (f(\cdot, x(\cdot), x'(\cdot), \varepsilon), C[x, \varepsilon], D[x, \varepsilon]),$$

so that problem (23) is equivalent to the abstract equation $Lx = \varepsilon N(x, \varepsilon)$.

Now, let us define $\gamma(t) = 1$ and $\delta(t) = e^{-at}$ for all $t \in [0,1]$ and assume conditions

$$\Lambda[\gamma] = \Theta[\gamma] = 0, \tag{24}$$

and

$$\Lambda[\delta] \neq 0 \quad \text{and} \quad \Theta[\delta] \neq 0. \tag{25}$$

Then it is clear that

$$\mathrm{Ker}(L) = \{x \in X : x \text{ is constant on } [0,1]\}.$$

On the other hand, the general solution of the equation $x'' + ax' = h(t)$ is given by

$$x(t) = c_1\gamma(t) + c_2\delta(t) + x_p(t) \quad \text{where} \quad x_p(t) := \int_0^t \frac{1}{a}(1 - e^{a(s-t)})h(s)ds.$$

Then, the associated linear non-homogeneous problem, with $h \in C([0,1])$, c, $d \in \mathbb{R}$, is

$$x'' + ax' = h(t),\ \Lambda[x] = c,\ \Theta[x] = d, \tag{26}$$

which is equivalent, in view of (24), to the existence of $c_2 \in \mathbb{R}$ such that

$$c_2\Lambda[\delta] + \Lambda[x_p] = c,\ c_2\Theta[\delta] + \Theta[x_p] = d. \tag{27}$$

It follows now from (25) and (27) that (26) is solvable if and only if

$$\Lambda[\delta]d = \Theta[\delta]c + \Lambda[\delta]\Theta[x_p] - \Theta[\delta]\Lambda[x_p],$$

so that L is a Fredholm mapping of index zero, and, for any projector $Q : Z \to Z$ such that $\mathrm{Im}(L) = \mathrm{Ker}(Q)$, the equation $Q(N(c_0, 0)) = 0$ with $c_0 \in \mathbb{R}$ is equivalent to

$$F(c_0) := \Theta[\delta]C[c_0, 0] - \Lambda[\delta]D[c_0, 0] + \Lambda[\delta]\Theta\left[\sigma\right] - \Theta[\delta]\Lambda\left[\sigma\right] = 0, \tag{28}$$

where

$$\sigma(t) := \int_0^t \frac{1}{a}(1 - e^{a(s-t)})f(s, c_0, 0, 0)ds.$$

Hence, as a direct consequence of Theorem 1 we obtain the following result.

Theorem 12. *For each $c_0 \in \mathbb{R}$ such that*

$$F(c_0) = 0 \quad \text{and} \quad F'(c_0) \neq 0,$$

where F is given by (28), there exists $\varepsilon_0 \in (0, \varepsilon_1]$ such that, for $0 < |\varepsilon| < \varepsilon_0$, the problem (23) has a unique solution $x(t, \varepsilon)$ such that $\lim_{\varepsilon \to 0} x(t, \varepsilon) = c_0$ uniformly in $t \in [0, 1]$.

4.2. **Problems with linear nonlocal boundary conditions.** Consider now for $a \in \mathbb{R}$, $a \neq 0$, the problem

$$x'' + ax' = \varepsilon f(t, x, x', \varepsilon), \quad x'(0) = \alpha[x], \quad x'(1) = \beta[x], \tag{29}$$

where α, β are continuous linear functionals given by the Riemann-Stieltjes integrals

$$\alpha[x] = \int_0^1 x(s)\, dA(s), \quad \beta[x] = \int_0^1 x(s)\, dB(s),$$

with the functions A, $B : [0, 1] \to \mathbb{R}$ of bounded variation. The problem (29) with $a = 0$ has been studied recently in [24]. Note that (29) is a particular instance of problem (23) setting $\Lambda[x] = x'(0) - \alpha[x]$, $\Theta[x] = x'(1) - \beta[x]$, $C[x, \varepsilon] \equiv 0$ and $D[x, \varepsilon] \equiv 0$.

Assume that

$$f : [0, 1] \times \mathbb{R}^2 \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}, \quad (t, u_0, u_1, \varepsilon) \quad \mapsto \quad f(t, u_0, u_1, \varepsilon),$$

is continuous and such that, for $k = 0, 1$, $\frac{\partial f}{\partial u_k}$ exists and is continuous, and moreover

$$\alpha[\gamma] = \beta[\gamma] = 0, \tag{30}$$

and

$$\alpha[\delta]+a\neq 0 \quad \text{and} \quad \beta[\delta]+ae^{-a}\neq 0. \tag{31}$$

Clearly, (30) and (31) are the equivalent to conditions (24) and (25). Notice also that (30) implies that problem (29) is at resonance, that is, the associated linear problem

$$x''+ax'=0, \quad x'(0)=\alpha[x], \quad x'(1)=\beta[x],$$

admits nontrivial solutions.

Let us introduce the notation:

$$k(t,s)=\begin{cases}\dfrac{1}{a}(1-e^{a(s-t)}), & 0\le s\le t\le 1,\\ 0, & 0\le t\le s\le 1,\end{cases}$$

$$\mathcal{K}_A(s)=\int_0^1 k(t,s)\,dA(t),\ \mathcal{K}_B(s)=\int_0^1 k(t,s)\,dB(t),$$

and

$$\eta(s)=\frac{\mathcal{K}_A(s)}{\alpha[\delta]+a}-\frac{\mathcal{K}_B(s)-e^{a(s-1)}}{\beta[\delta]+ae^{-a}}. \tag{32}$$

Then the averaged function given by (28) is (up to a multiplicative constant different from zero)

$$F:\mathbb{R}\to\mathbb{R}, \quad c\mapsto F(c):=\int_0^1 \eta(s)f(s,c,0,0)\,ds. \tag{33}$$

Then $F\in C^1(\mathbb{R},\mathbb{R})$ and, for all $c\in\mathbb{R}$,

$$F'(c)=\int_0^1 \eta(s)\frac{\partial f}{\partial x}(s,c,0,0)\,ds.$$

Hence, as a direct consequence of Theorem 1 we obtain the following result.

Theorem 13. *Assume that (30) and (31) hold. Then, for each $c_0\in\mathbb{R}$ such that*

$$F(c_0)=0 \quad \textit{and} \quad F'(c_0)\neq 0,$$

where F is given by (33), there exists $\varepsilon_0\in(0,\varepsilon_1]$ such that, for $0<|\varepsilon|<\varepsilon_0$, the problem (29) has a unique solution $x(t,\varepsilon)$ such that $\lim_{\varepsilon\to 0} x(t,\varepsilon)=c_0$ uniformly in $t\in[0,1]$.

4.2.1. *An application to a tubular chemical reactor.* Given α_1 and α_2 linear continuous functionals on $C([0,1])$ let us consider the nonlocal boundary value problem

$$x''-\lambda x'=-\mu\lambda(b(t)-x)e^{x},\ x'(0)=\lambda x(0)+\alpha_1[x], x'(1)=\alpha_2[x], \tag{34}$$

where $\lambda>0$ is the Peclet number, $\mu>0$ is the Damkohler number and $b:[0,1]\to(0,+\infty)$ is a continuous function representing the dimensionless adiabatic temperature rise. Problem (34) models the steady states of a chemical reactor in a tube of unitary length being $x(t)$ the temperature at a distance t along the tube, see [11].

Setting

$$\alpha[x]:=\lambda x(0)+\alpha_1[x] \quad \text{and} \quad \beta[x]:=\alpha_2[x],$$

since they are linear continuous functionals on $C([0,1])$, it follows from the Riesz representation theorem, [21], the existence of A and B, functions of bounded variation, such that α and β can be expressed as the Riemann-Stieltjes integrals

$$\alpha[x] = \int_0^1 x(s)dA(s) \quad \text{and} \quad \beta[x] = \int_0^1 x(s)dB(s).$$

So, problem (34) is a particular case of (29). By considering the Damkohler number μ as a small parameter we obtain the following result.

Corollary 14. *Let us assume that α_1 and α_2 satisfy*

$$\alpha_1[\gamma] + \lambda = \alpha_2[\gamma] = 0, \tag{35}$$

and

$$\alpha_1[\delta] \neq 0 \quad \text{and} \quad \alpha_2[\delta] - \lambda e^{\lambda} \neq 0, \tag{36}$$

where $\gamma(t) = 1$ and $\delta(t) = e^{\lambda t}$ and moreover that

$$\int_0^1 \eta(s)ds \neq 0, \tag{37}$$

where η is given by (32).

Then there exists $\mu_0 > 0$ such that for $0 < \mu < \mu_0$ there is a unique solution $x(t,\mu)$ of problem (34) and

$$\lim_{\mu\to 0^+} x(t,\mu) = \frac{1}{\int_0^1 \eta(s)\,ds}\int_0^1 \eta(s)b(s)\,ds \quad \text{uniformly in } t \in [0,1].$$

Proof. It is clear that assumptions (35) and (36) are equivalent to conditions (30) and (31). On the other hand, the averaged function F corresponding to problem (34), as given by (33), is

$$F(c) = \int_0^1 \eta(s)\lambda(b(s) - c)e^c\,ds.$$

Then, it its easy to show that for $c_0 = \dfrac{1}{\int_0^1 \eta(s)\,ds}\displaystyle\int_0^1 \eta(s)b(s)\,ds$ we have

$$F(c_0) = 0 \quad \text{and} \quad F'(c_0) = -\lambda e^{c_0}\left(\int_0^1 \eta(s)\,ds\right) \neq 0.$$

Therefore the result follows from Theorem 13. □

Remark 15. Let $\tau(t) = t$. A direct computation shows that

$$\int_0^1 \eta(s)ds = \frac{1-\alpha_1[\tau]}{\lambda\alpha_1[\delta]} - \frac{1-\alpha_2[\tau]}{\lambda(\alpha_2[\delta] - \lambda e^{\lambda})},$$

so condition (37) is satisfied for instance if $\alpha_1[\tau] = 1$ and $\alpha_2[\tau] \neq 1$ or viceversa.

Remark 16. In [11] the authors deal with the existence and localization of positive solutions of (34). It follows from Corollary 14, that if

$$c_0 = \frac{1}{\int_0^1 \eta(s)\,ds}\int_0^1 \eta(s)b(s)\,ds > 0$$

then (34) has a unique positive solution provided μ is small enough.

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References

[1] R. A. Bonic, *Linear Functional Analysis*, Gordon and Breach, New York, 1969.

[2] A. Buică and J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, *Bull. Sci. Math.* **128** (2004), 7–22.

[3] J.A. Cid, G. Propst and M. Tvrdý, On the pumping effect in a pipe/tank flow configuration with friction, *Phys. D* **273-274** (2014), 28–33.

[4] J.A. Cid, G. Infante, M. Tvrdý and M. Zima, A topological approach to periodic oscillations related to the Liebau phenomenon, *J. Math. Anal. Appl.* **423** (2015), 1546–1556.

[5] J.A. Cid, G. Infante, M. Tvrdý and M. Zima, New results for the Liebau phenomenon via fixed point index, *Nonlinear Anal. RWA* **35** (2017), 457–469.

[6] J. Cronin, *Fixed Points and Topological Degree in Nonlinear Analysis*, American Mathematical Society, Providence RI, 1964.

[7] R. E. Gaines and J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Mathematics 568, Springer-Verlag, Berlin, 1977.

[8] J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Applied Mathematical Sciences 42, Springer-Verlag, New York, 1983.

[9] J. K. Hale, *Oscillations in Nonlinear Systems*, McGraw-Hill, New York, 1963.

[10] J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York, 1969. Second Ed., Krieger Publishing Company, Huntington, N.Y., 1980.

[11] G. Infante, P. Pietramala and M. Tenuta, Existence and localization of positive solutions for a nonlocal BVP arising in chemical reactor theory, *Commun. Nonlinear Sci. Numer. Simulat.* **19** (2014), 2245–2251.

[12] F. Liao, Periodic solutions of Liebau-type differential equations, *Appl. Math. Lett.* **69** (2017), 8–14.

[13] G. Liebau, Über ein ventilloses Pumpprinzip, *Naturwissenschaften* **41** (1954), 327.

[14] J. Llibre, D. D. Novaes and M. A. Teixeira, Higher order averaging theory for finding periodic solutions via Brouwer degree, *Nonlinearity* **27** (2014), 563–583.

[15] J. Llibre and D.D. Novaes, Improving the averaging theory for computing periodic solutions of the differential equations, *Z. Angew. Math. Phys.* **66** (2015), 1401–1412.

[16] J. Mawhin, Periodic solutions in some perturbed differential equations, *Boll. Un. Mat. Ital.* (4) **11** (1975), 299–305.

[17] J. Mawhin, Functional analysis and boundary value problems, in *Studies in Ordinary Differential Equations*, J.K. Hale (ed.), Mathematical Association of America, Washington, 1977, 128–168.

[18] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Regional Conference Series in Mathematics 40, American Mathematical Society, Providence, 1979.

[19] J. Mawhin, Periodic solutions in the golden sixties : the birth of a continuation theorem, in *Ten Mathematical Essays on Approximation in Analysis and Topology*, J. Ferrera, J. López-Gómez and F. Ruiz del Portal eds., Elsevier, Amsterdam, 2005, 199–214.

[20] G. Propst, Pumping effects in models of periodically forced flow configurations, *Phys. D* **217** (2006), 193–201.

[21] F. Riesz, Sur les opérations fonctionnelles linéaires, *C. R. Acad. Sci. Paris* **149** (1909), 974–977.

[22] N. Rouche and J. Mawhin, *Équations différentielles ordinaires. II. Stabilité et solutions périodiques*, Masson, Paris, 1973. English transl. *Ordinary Differential Equations. Stability and Periodic Solutions*, Surveys and Reference Works in Mathematics 5, Pitman, Boston, 1980.
[23] J.A. Sanders, F. Verhulst and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, Applied Mathematical Sciences 59, Springer-Verlag, New York, 2007.
[24] K. Szymańska-Dębowska and M. Zima, A topological degree approach to a nonlocal Neumann problem for a system at resonance, *J. Fixed Point Theory Appl.* **21** (2019), Art. 67, 14 pp.
[25] P. Torres, Existence and stability of periodic solutions of a Duffing equation by using a new maximum principle, *Mediterr. J. Math.* **1** (2004), 479–486.
[26] P. J. Torres, *Mathematical Models with singularities - A Zoo of Singular Creatures*, Atlantis Press, Paris, 2015.
[27] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext, Springer-Verlag, Berlin-Heidelberg, 1996.
[28] F. Wang, J.A. Cid, S. Li and M. Zima, Lyapunov stability of periodic solutions of Brillouin type equations, *Appl. Math. Lett.* **101** (2020), 106057, 7 pages.
[29] F. Wang, J.A. Cid and M. Zima, Lyapunov stability for regular equations and applications to the Liebau phenomenon, *Discrete Contin. Dyn. Syst.* **38** (2018), 4657–4674.
[30] E. Zeidler, *Nonlinear Functional Analysis and its Applications. I. Fixed-Point Theorems*, Springer, New York, 1986.

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