A note on fixed points theorems for $T-$ monotone operators. *

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Abstract

This paper contains two contributions of the theory of T–monotone operators introduced by Chen. First, we prove a new fixed point theorem for a discontinuous T–monotone mapping. After, we use this theory to obtain the solution of a classical continuous problem, for which the usual iterative methods fail.

1 Introduction

The concept of a T–monotone operator A, that is, $A+T$ is nondecreasing, was introduced by Chen in [1], where he proves that the classical monotone method for nondecreasing and condensing maps is valid to this much larger class of operators. Later, Syau gave in [2] some new fixed points theorems for T–monotone operators.

In section 2 we use the generalized iterative technique for discontinuous monotone operators of Heikkilä and Lakshmikantham [3] to obtain the existence of extremal fixed points for discontinuous T–monotone operators improving some results of [2].

In section 3 we present an example which shows that the theory of T–monotone operators is applicable to an initial value problem for a first order differential equation, even when the classical monotone method and the Picard iterates fail to converge to its solution.

2 A new fixed point theorem

Let E be a real Banach space ordered by a cone K, i.e. $x \leq y$ if and only if $y - x \in K$. The cone K is regular if every nondecreasing sequence which is order bounded from above is already convergent. We point out that the cone of almost everywhere nonnegative functions in the space $L^p(\Omega)$, $1 \leq p < \infty$, with Ω

an open and bounded set of $\mathbb{R}^n$, is regular. For more examples of Banach spaces with regular cone see section 5.8 in [3].

Let $A : D \subset E \to E$ and $T : E \to E$ be two operators. We say that A is a T-monotone operator if

$$Ax - Ay \geq Ty - Tx, \quad x \geq y, \quad x, y \in D,$$

that is, if operator $A + T$ is nondecreasing in D.

Assume now that there is an operator $T : E \to E$ satisfying:

(C1) T is nondecreasing in E, and

(C2) There exists $\lambda \in (0, 1]$ such that $\lambda I + T : E \to E$ is one to one and $(\lambda I + T)^{-1}$ is nondecreasing in E.

Note that we do not impose to operator A and T to be continuous.

On the other hand, when operator T is linear, hypothesis $(C1)$ and $(C2)$ are conditions $(T1)$ and $(T2)$ in [1].

The proof of the next lemma, which is fundamental in our work, is similar to the ones given in lemmas 1, 3 and 4 of [1] and we omit it.

Lemma 2.1 *Let $u_0, v_0 \in E$, $u_0 \leq v_0$, $A : [u_0, v_0] \to E$ a $T-$ monotone operator such that $u_0 \leq Au_0$ and $Av_0 \leq v_0$. Moreover suppose that operator T satisfies conditions* (C1) *and* (C2) *for some $\lambda \in (0, 1]$ fixed. Then the following operator*

$$S := (\lambda I + T)^{-1}(\lambda A + T) : [u_0, v_0] \to E, \tag{2.1}$$

satisfies that:

i) $x \in [u_0, v_0]$ *and* $S\,x = x$ *if and only if* $x \in [u_0, v_0]$ *and* $A\,x = x$.

ii) S *is nondecreasing in* $[u_0, v_0]$.

iii) $S([u_0, v_0]) \subset [u_0, v_0]$.

Next theorem is the main result of this section and improves theorems 3.1 and 3.2 in [2].

Theorem 2.2 *Let K be a regular cone, $u_0, v_0 \in E$, $u_0 \leq v_0$, $A : [u_0, v_0] \to E$ be T-monotone with $u_0 \leq Au_0$ and $Av_0 \leq v_0$, and suppose that T satisfies* (C1) *and* (C2).

Then A has the minimal fixed point x_* and the maximal fixed point x^* in $[u_0, v_0]$, which are characterized by the following properties:

$$x_* = \min\{x \in [u_0, v_0] : Ax \leq x\}, \ x^* = \max\{x \in [u_0, v_0] : Ax \geq x\}. \tag{2.2}$$

Moreover defining the sequences

$$u^0 = u_0, \ u^m = \lim_{n\to\infty} S^n u^{m-1} \quad \text{and} \quad v^0 = v_0, \ v^m = \lim_{n\to\infty} S^n v^{m-1},$$

for all $m \in \mathbb{N}$, where S is the operator defined in (2.1), we have that:

a) $u^0 \leq u^m \leq u^{m+1} \leq x_* \leq x^* \leq v^{m+1} \leq v^m \leq v^0$ *for all $m \in \mathbb{N}$.*

b) $x_* = u^m$ *if and only if* $u^m = Au^m$. *This holds if S is left continuous at u^m.*

c) $x^* = v^m$ *if and only if* $v^m = Av^m$. *This holds if S is right continuous at v^m.*

d) If S is left continuous, then $x_* = u^1 = \lim_{n\to\infty} S^n u^0$.

e) If S is right continuous, then $x^* = v^1 = \lim_{n\to\infty} S^n v^0$.

Proof. By lemma 2.1 *ii), iii)* we have that operator $S : [u_0, v_0] \to [u_0, v_0]$ is nondecreasing. Thus, since K is regular, we have that $\{Sx_n\}_{n\in\mathbb{N}}$ is a convergent sequence whenever $\{x_n\}_{n\in\mathbb{N}}$ is a monotone one. Then theorem 1.2.2 in [3] ensures the existence of the minimal fixed point x_* and the maximal fixed point x^* of S in $[u_0, v_0]$, which are characterized by

$$x_* = \min\{x \in [u_0, v_0] : Sx \leq x\}, \ x^* = \max\{x \in [u_0, v_0] : Sx \geq x\}. \tag{2.3}$$

From lemma 2.1, *i)* it follows then that x_* and x^* are the minimal and the maximal fixed points of A in $[u_0, v_0]$, respectively. Moreover, since $Sx \leq (\geq) x$ if and only if $Ax \leq (\geq) x$, from (2.3) we obtain (2.2).

Finally, claims *a)–e)* follow by applying corollary 1.2.2 in [3] to operator S and taking into account that, by lemma 2.1, *i)*, $x = Sx$ if and only if $x = Ax$. □

Now, we deduce the following particular case of theorem 2.2, which gives us a useful form to apply the previous existence result and imposes a one – sided Lipschitz condition in operator A.

Corollary 2.3 *Let K be a regular cone, $u_0, v_0 \in E$, $u_0 \leq v_0$, $A : [u_0, v_0] \to E$ be an operator for which there is a real constant $M \geq 0$ such that*

$$Ax - Ay \geq M(y - x), \quad x \geq y, \quad x, y \in [u_0, v_0], \tag{2.4}$$

and such that $u_0 \leq Au_0$ and $Av_0 \leq v_0$.

Then A has the minimal fixed point x_* and the maximal fixed point x^* in $[u_0, v_0]$, which are characterized by (2.2).

Moreover, defining $u^1 = \lim_{n\to\infty} u_n$ and $v^1 = \lim_{n\to\infty} v_n$ where

$$u_n = \frac{1}{M+1} Au_{n-1} + \frac{M}{M+1} u_{n-1} \quad \text{and} \quad v_n = \frac{1}{M+1} Av_{n-1} + \frac{M}{M+1} v_{n-1},$$

for all $n \in \mathbb{N}$, we have that:

a) $u_0 \leq u_n \leq u_{n+1} \leq u^1 \leq x_* \leq x^* \leq v^1 \leq v_{n+1} \leq v_n \leq v_0$ *for all $n \in \mathbb{N}$.*

b) $x_* = u^1$ *if and only if $u^1 = Au^1$. This holds if A is left continuous at u^1.*

c) $x^* = v^1$ *if and only if $v^1 = Av^1$. This holds if A is right continuous at v^1.*

Proof. Condition (2.4) says us that A is T-monotone, defining $Tx \equiv Mx$ for all $x \in E$. Furthermore T satisfies assumptions (C1) and (C2) whit $\lambda = 1$. Then, taking into account that operator S defined in (2.1) is given, in this particular situation, by

$$Sx = \frac{1}{M+1} Ax + \frac{M}{M+1} x \quad \text{for all } x \in [u_0, v_0],$$

that S is nondecreasing and that the lateral continuity of S is equivalent to the lateral continuity of A, the assertions follow from theorem 2.2. $\square$

3 A nontrivial example

In this section, we use the theory of T–monotone operators to approximate the unique solution of an initial value problem where the classical Picard iterates [4] do not converge and the monotone method coupled with lower and upper solutions [5] is not applicable.

To this end, consider the Cauchy problem

$$x'(t) = f(t, x(t)), \text{ for all } t \in I = [0,1], \quad x(0) = 0, \tag{3.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t,x) = \begin{cases} -2t, & \text{if } \ t \in [0,1],\ x \geq t^2, \\ 2t - \dfrac{4x}{t}, & \text{if } \ t \in (0,1],\ 0 \leq x \leq t^2, \\ 2t, & \text{if } \ t \in [0,1],\ x \leq 0. \end{cases}$$

Although f is continuous, and therefore its study does not require the specific development of fixed points theorems for discontinuous operators, we have chosen this example due to the fact that problem (3.1) is a classical initial value problem for which the sequence of successive Picard approximations does not converge to a solution. Moreover, since $f(t, \cdot)$ is nonincreasing we have that if problem (3.1) has a solution then it is unique (see [4], page 41).

On the other hand, the functions

$$u_0(t) = -t^2 \quad \text{and} \quad v_0(t) = t^2 \quad \text{for all } t \in I,$$

are a lower and an upper solution of (3.1) respectively, and then theorem 1.1.4 in [5] ensures that there exists a solution x of (3.1) such that $u_0(t) \leq x(t) \leq v_0(t)$ for all $t \in I$. Therefore the unique solution of problem (3.1) lies between u_0 and v_0.

We remark that the classical monotone iterative technique exposed at theorem 1.2.1 in [5] is not applicable to problem (3.1) since it does not exist $M \geq 0$ such that

$$f(t,x) - f(t,y) \geq -M(x-y) \quad \text{for all } t \in I \text{ and } -t^2 \leq y \leq x \leq t^2.$$

Clearly, the solutions of (3.1) are the fixed points of operator $A : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ defined by

$$Ax(t) = \int_0^t f(s, x(s))ds \quad \text{for all } t \in I.$$

It is easy to verify that the set $K = \{cv_0 \ : \ c \geq 0\}$, is a regular cone in $\mathcal{C}(I)$. For given $x, y \in \mathcal{C}(I)$, the partial ordering induced by K in $\mathcal{C}(I)$, which we will denote by $\preceq$, is the following:

$$x \preceq y \text{ if and only if there exists } c = c(x,y) \geq 0 \text{ such that } y - x = cv_0.$$

It is obvious that $u_0 \preceq v_0$ and that

$$[u_0, v_0] := \{x \in \mathcal{C}(I) : u_0 \preceq x \preceq v_0\} = \{cv_0 : c \in [-1,1]\}.$$

Since $Au_0 = v_0$ and $Av_0 = u_0$ we have that

$$u_0 \preceq Au_0 \quad \text{and} \quad Av_0 \preceq v_0.$$

Now, define $T\,x \equiv 2\,x$ for all $x \in C(I)$. We are going to prove that operator A is T-monotone in $[u_0, v_0]$. Let $x, y \in [u_0, v_0]$ such that $x \succeq y$. Then $y = c_1\,v_0$ and $x = c_2\,v_0$, with $c_1, c_2 \in [-1,1]$, $c_1 \leq c_2$ and we have that:

i) If $c_1, c_2 \in [-1, 0]$, then $Ay(t) - Ax(t) = 0 \quad$ for all $t \in I$.

ii) If $c_1 \in [-1, 0]$ and $c_2 \in (0, 1]$, then $Ay(t) - Ax(t) = 2x(t) \quad$ for all $t \in I$.

iii) If $c_1, c_2 \in (0, 1]$, then $Ay(t) - Ax(t) = 2(x(t) - y(t)) \quad$ for all $t \in I$.

Therefore it holds that

$$Ax - Ay \succeq 2(y - x), \quad \text{for all } x, y \in [u_0, v_0] \text{ such that } x \succeq y.$$

As consequence, operator A satisfies (2.4) for $M = 2$. From the continuity of function f respect to the second variable, we deduce that operator A is continuous too. Now corollary 2.3 ensures us that the sequences

$$u_n = \frac{1}{3}Au_{n-1} + \frac{2}{3}u_{n-1} \quad \text{and} \quad v_n = \frac{1}{3}Av_{n-1} + \frac{2}{3}v_{n-1} \quad \text{for all } n \in \mathbb{N},$$

converge to the minimal fixed point x_* and to the maximal fixed point x^* of A in $[u_0, v_0]$, respectively. It is easy to verify that

$$u_1(t) = -\frac{1}{3}t^2, \quad u_2(t) = \frac{1}{9}t^2 \quad \text{and} \quad u_n(t) = \frac{1}{3}t^2 \quad \text{for all } n \geq 3,$$

$$v_n(t) = \frac{1}{3}t^2 \quad \text{for all } n \geq 1,$$

and then

$$x_*(t) = x^*(t) = \frac{1}{3}t^2 \quad \text{for all } t \in I,$$

is the unique fixed point of A and therefore it is also the unique solution of (3.1).

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