On a class of singular Sturm-Liouville periodic boundary value problems*

Alberto Cabada† and J. Ángel Cid‡

† *Departamento de Análise Matemática, Facultade de Matemáticas, Universidade de Santiago de Compostela, 15782, Santiago de Compostela, Spain.*
E–mail: alberto.cabada@usc.es.

‡*Departamento de Matemáticas, Ed. B3, Campus Las Lagunillas, Universidad de Jaén, 23071, Jaén, Spain.*
E–mail: angelcid@ujaen.es.

Abstract

Keeping in mind the singular model for the periodic oscillations of the axis of a satellite in the plane of the elliptic orbit around its center of mass, we give sufficient conditions for the solvability of a class of singular Sturm-Liouville equations with periodic boundary value conditions. To this end, under a suitable change of variables, we present a new existence result for problems defined in the real half-line.

Keywords: Singular Sturm-Liouville problem, periodic solution, solutions on unbounded intervals.
2010 MSC: 34B15, 34C25, 34B40.

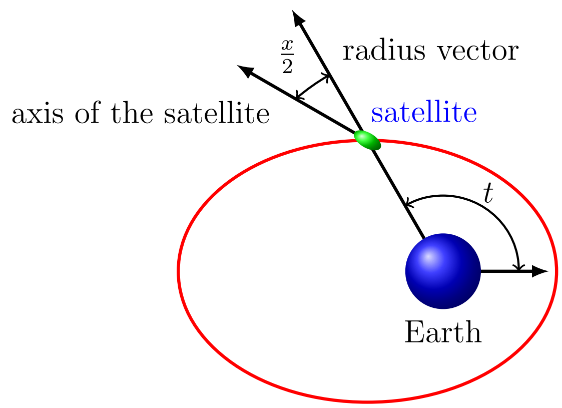
1 Introduction

The boundary value problem

$$\begin{cases} (1+e\cos(t))x'' - 2e\sin(t)x' + \lambda\sin(x) = 4e\sin(t), & t\in[0,2\pi], \\ x(0)=x(2\pi), \qquad x'(0)=x'(2\pi), \end{cases} \tag{1.1}$$

was introduced by Beletskii [4, 5, 6] as a model for the periodic oscillations of the axis of a satellite in the plane of the elliptic orbit around its center of mass, where $0 \le e < 1$ is the eccentricity of the ellipse and $|\lambda| \le 3$ is a parameter related with the inertia of the satellite.

*This work was partially supported by Ministerio de Educación y Ciencia, Spain, project MTM2007-61724.



From the mathematical point of view it is interesting to study for which values of the parameters in the (e, λ)-plane the problem (1.1) has a solution. The solvability of (1.1) seems to be studied for the first time in [17, 21, 22], but in 1985 Petryshyn and Yu made a major step by establishing the existence of solution for (1.1) when

$$0 \leq e\,\pi < 2\,|\lambda| < 1 - (8\sqrt{2} + 3)\,e,$$

by using the degree theory for A-proper mappings.

In 1988 Hai proves for $|e| < 1$ and $\lambda \in \mathbb{R}$ the existence of a periodic solution as a minimum in a certain ball of an associated functional [14] and the existence of an odd periodic solution by using a monotone iterative scheme [15]. Also in 1988 Mawhin [18] proves the existence of an odd periodic solution for $|e| < 1$ and $\lambda \in \mathbb{R}$ and the existence of a second solution for a suitable restricted region of the parameters. Two years later Hai improves the multiplicity result of Mawhin (see [16]) by proving that for all the values $|e| < 1$ and $\lambda \in \mathbb{R}$ there exist at least two solutions of (1.1) not differing by a multiple of 2π. He obtained these two solutions as different critical points of an associated functional. On the other hand, the stability of the solutions of (1.1) has been studied in [19].

If we multiply the differential equation of problem (1.1) by $(1 + e\cos(t))$ then it can be rewritten as

$$\begin{cases} ((1 + e\cos(t))^2 x')' = 4e(1 + e\cos(t))\sin(t) - \lambda(1 + e\cos(t))\sin(x), \;\; t \in [0, 2\pi], \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{cases} \tag{1.2}$$

which is in the Sturm-Liouville form $(p(t)x')' = f(t, x)$ with

$$p(t) = (1 + e\cos(t))^2,$$

and

$$f(t, x) = 4e(1 + e\cos(t))\sin(t) - \lambda(1 + e\cos(t))\sin(x).$$

Problem (1.1) is said to be regular when $0 \leq e < 1$ and singular for $e = 1$ (see [9, 10]). This is due to the fact that in the last case the coefficient of the second order derivative vanishes at $t = \pi$. Moreover problem (1.2) becomes a singular Sturm-Liouville problem whenever $e = 1$, since $\int_0^\pi \frac{1}{p(s)} ds = +\infty$ (see [12, 13]). This fact makes the case $e = 1$ interesting to deal with.

Now, keeping in mind problem (1.2) with $e = 1$, we are going to study the singular Sturm-Liouville periodic boundary value problem

$$\begin{cases} (p(t)x')' = f(t,x), & t \in [0, 2T], \\ x(0) = x(2T), & x'(0) = x'(2T). \end{cases} \tag{1.3}$$

where the nonlinearities p and f shall satisfy some suitable symmetry conditions, $p(t) > 0$ for all $t \in [0, T)$ and $\int_0^T \frac{1}{p(s)} ds = +\infty$. Our assumptions will allow us to search a solution of problem (1.3) as the odd extension of a solution of the Dirichlet problem

$$\begin{cases} (p(t)\, x')' = f(t,x), & t \in [0, T], \\ x(0) = 0 = x(T), & \lim\limits_{t \to T^-} p(t)\, x'(t) \text{ exists.} \end{cases} \tag{1.4}$$

To deal with this Dirichlet problem we perform the standard Liouville transformation on the independent variable

$$\tau(t) = \int_0^t \frac{ds}{p(s)}, \tag{1.5}$$

which, due to the singularity of $p(t)$, leads us to the half-line boundary value problem

$$\begin{cases} y'' = g(\tau, y), & \tau \in [0, +\infty), \\ y(0) = 0 = y(+\infty), & y'(+\infty) \text{ exists,} \end{cases} \tag{1.6}$$

with g a suitable function related to f.

In section 2 we shall prove the existence of a solution of problem (1.6) under conditions that, as far as we know, are not covered by previously ones considered in the literature for boundary value problems in infinite intervals (see [1, 2, 3, 7, 8, 23] and references therein). Finally, in section 3 we shall give an application of our main result to problem (1.2) in the singular case $e = 1$.

2 Main result

This section is devoted to prove the existence of a nontrivial odd solution of problem (1.3). To this end, we assume the following list of assumptions:

(p0) $p : [0, 2T] \to \mathbb{R}$ is continuous, $p(T - t) = p(T + t)$ for all $t \in [0, T]$, $p(t) > 0$ for all $t \in [0, T)$ and $\int_0^T \frac{1}{p(s)} ds = +\infty$.

(p1) There is $\alpha > 1$ such that

$$\lim_{n\to\infty} \int_{n/\alpha}^{n} (n-s)\, p(t(s))\, ds = +\infty,$$

with $t : [0,\infty) \to [0,T]$ given by $t(s) := \tau^{-1}(s)$ for all $s \in [0,\infty)$, and τ defined on (1.5).

(f0) $f : [0, 2T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

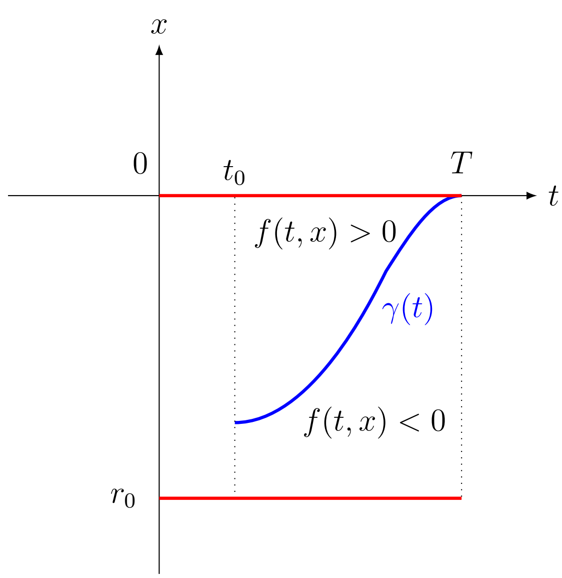
$$f(T-t,-x) = -f(T+t,x) \quad \text{for all } t \in [0,T] \text{ and all } x \in \mathbb{R}.$$

(f1) There exist $t_0 \in (0,T)$, a constant $r_0 < 0$ and a nondecreasing continuous curve $\gamma : [t_0,T] \to (-\infty,0]$, with $r_0 < \gamma(t) < 0$ for all $t \in (t_0,T)$ and $\gamma(T) = 0$, such that:

(i) $f(t,\gamma(t)) = 0$ for all $t \in [t_0,T]$,

(ii) $f(t,0) \geq 0$ for all $t \in (0,t_0]$. Moreover $f(t,x) > 0$ for all $t \in (t_0,T]$ and $\gamma(t) < x \leq 0$,

(iii) $f(t,r_0) \leq 0$ for all $t \in (0,t_0]$. Moreover $f(t,x) < 0$ for all $t \in (t_0,T]$ and $r_0 \leq x < \gamma(t)$.



Remark 2.1 *It is clear that assumptions (p0) and (f0) imply, respectively, that $p(T) = 0$ and $f(T,0) = 0$.*

On the other hand, if f is a continuously differentiable function in a neighborhood of $(T,0)$, with $f(T,0) = 0$ and $\frac{\partial f}{\partial x} f(T,0) \neq 0$, then by the Implicit Function Theorem there exist $t_1 \in (0,T)$ and a continuously differentiable curve $\gamma : (t_1,T] \to \mathbb{R}$ such that $\gamma(T) = 0$, $f(t,\gamma(t)) = 0$ for all $t \in (t_1,T]$ and $\gamma'(t) = -\frac{\partial f}{\partial t}(t,\gamma(t))/\frac{\partial f}{\partial x}(t,\gamma(t))$.

In consequence, if moreover

$$\frac{\partial f}{\partial t}(0,T)/\frac{\partial f}{\partial x}(0,T) < 0,$$

then there exists $t_1 \le t_2 < T$ *such that* γ *is increasing on* $(t_2, T]$ *and* $\gamma(t) < 0$ *for all* (t_2, T).

Definition 2.1 *By a solution of problem (1.3) we mean a function*

$$x \in C([0, 2T]) \bigcap C^1([0,T) \cup (T, 2T])$$

with $p(t)\, x' \in C^1([0, 2T])$ *and that satisfies the differential equation and the boundary conditions.*

Now we present our main result.

Theorem 2.2 *If assumptions (p0), (p1), (f0) and (f1) hold, then problem (1.3) has a nontrivial odd solution (with respect to* $t = T$*) which moreover satisfy* $r_0 \le u(t) \le 0$ *for all* $t \in (0,T)$ *and* $u(t) < \gamma(t) < 0$ *for all* $t \in (\bar{t}, T)$, *for some* $\bar{t} \in (t_0, T)$.

Proof. Conditions (p0) and (f0) imply that the odd extension of a solution of problem (1.4) is a solution of (1.3). By using the change in the independent variable τ given in (1.5), which is, by (p0), an increasing homeomorphism from $[0,T)$ onto $[0,\infty)$, we obtain that x is a solution of problem (1.4) if and only if $y(\tau(t)) := x(t)$ is a solution of problem (1.6) with

$$g(\tau, y) = p(t(\tau)) f(t(\tau), y),$$

and $t(\tau)$ given in (p1).

Now, by denoting $\tau_0 \equiv \tau(t_0)$, we divide the proof into several steps.

Claim 1.- For each $n \in \mathbb{N}$, $n \ge \tau_0$, *the Dirichlet problem*

$$\begin{cases} y''(\tau) = g(\tau, y(\tau)), \tau \in [0,n], \\ y(0) = 0 = y(n), \end{cases} \tag{2.1}$$

has a nontrivial solution $y_n \in [r_0, 0]$. *Moreover* $y_n(\tau) < 0$ *for all* $\tau \in (\tau_0, n)$.

From (p0) and (f1) it follows that

$$g(\tau, r_0) \le 0 \le g(\tau, 0) \quad \text{for all } \tau \in (0, \infty).$$

Hence it is clear that for each $n \in \mathbb{N}$ the functions

$$\alpha(\tau) = r_0 \quad \text{and} \quad \beta(\tau) = 0,$$

are a lower and an upper solution, respectively, for problem (2.1) with $\alpha \le \beta$. Therefore it is well-known (see [11, Theorem 4.2]) that problem (2.1) has a solution y_n with $r_0 \le y_n(\tau) \le 0$ for all $\tau \in [0, n]$.

Now, suppose that $y_n(\tau_n) = 0$ for some $\tau_n \in (\tau_0, n)$. Using condition (f1) and the definitions of g and τ_0, we have that $y_n''(\tau_n) > 0$, which contradicts the fact that y_n attains a local maximum at τ_n.

Claim 2.- There exists a constant $M > 0$ (independently of $n \in \mathbb{N}$) such that

$$\max\{\|y_n\|_\infty, \|y_n'\|_\infty, \|y_n''\|_\infty\} \leq M \quad \textit{for all } n \in \mathbb{N}.$$

Since $\|y_n\|_\infty \leq |r_0|$ for all $n \in \mathbb{N}$ and in view that y_n satisfies the differential equation $y'' = g(\tau, y)$ on $[0, n]$, it follows that also $\|y_n''\|_\infty \leq M_2$ for some constant $M_2 > 0$ independently of n. These two facts imply easily that y_n' must be also bounded independently of n.

Claim 3.- A subsequence of $\{y_n\}_{n\in\mathbb{N}}$ converges uniformly on compact sets to a function $y \in C^2[0,\infty)$ which satisfies $y(0) = 0$, $r_0 \leq y(\tau) \leq 0$ and $y''(\tau) = g(\tau, y(\tau))$ for all $\tau \in [0, +\infty)$.

Taking into account that y_n, y_n' and y_n'' are uniformly bounded, from the Ascoli-Arzelà theorem and by using a diagonal argument we obtain a subsequence of $\{y_n\}_{n\in\mathbb{N}}$ which converges to a function y uniformly on each compact set. Thus $y(0) = 0$, $r_0 \leq y(\tau) \leq 0$ and y satisfies the differential equation $y''(\tau) = g(\tau, y(\tau))$ for all $\tau \in [0, \infty)$.

Claim 4.- $\lim\limits_{\tau\to\infty} y'(\tau) = 0$ and there exists $\lim\limits_{\tau\to\infty} y(\tau)$. Moreover there is $\tau_1 \geq \tau_0$ such that $y(\tau) < \tilde{\gamma}(\tau) := \gamma(t(\tau)) < 0$ for all $\tau > \tau_1$.

Due to assumption $(f1)$, if for all $\tau \geq \tau_0$ function $y(\tau)$ is always above the curve $\tilde{\gamma}(\tau)$ then it would be convex on (τ_0, ∞), which is impossible. On the other hand, since $\tilde{\gamma}(\tau)$ is nondecreasing for all $\tau \geq \tau_0$, we have that $y(\tau)$ must be below the curve $\tilde{\gamma}(\tau)$ after some $\tau_1 \geq \tau_0$. Therefore for $\tau \geq \tau_1$ the function is concave and bounded, which imply that $\lim\limits_{\tau\to\infty} y(\tau)$ exists and $\lim\limits_{\tau\to\infty} y'(\tau) = 0$.

Claim 5.- $\lim\limits_{\tau\to\infty} y(\tau) = 0$.

To the contrary, suppose that $\lim\limits_{\tau\to\infty} y(\tau) = y_0 \in [r_0, 0)$. Now, for any $n \in \mathbb{N}$ fixed, let z_n be the unique solution of the Dirichlet problem

$$z''(\tau) = g(\tau, y(\tau)),\ \tau \in [0, n], \quad z(0) = z(n) = 0,$$

which is given by

$$z_n(\tau) = \int_0^n G_n(\tau, s)\, g(s, y(s))\, ds,$$

where G_n is the corresponding Green's function which explicit expression is

$$G_n(\tau, s) = \begin{cases} s\left(\frac{\tau}{n} - 1\right), \text{ if } 0 \leq s \leq \tau \leq n, \\ \\ \tau\left(\frac{s}{n} - 1\right), \text{ if } 0 \leq \tau \leq s \leq n. \end{cases} \tag{2.2}$$

Clearly, $w_n := y - z_n$ satisfies the following equalities:

$$w_n''(\tau) = 0,\ \tau \in [0, n], \quad w_n(0) = 0,\ w_n(n) = y(n),$$

or equivalently,

$$y(\tau) = \int_0^n G_n(\tau, s)\, g(s, y(s))\, ds + \frac{\tau}{n}\, y(n), \quad \text{for all } \tau \in [0, n].$$

In consequence, evaluating the previous expression at $\tau = n/\alpha$, we deduce that

$$y(n/\alpha) = \int_0^n G_n(n/\alpha, s)\, g(s, y(s))\, ds + \frac{1}{\alpha} y(n),$$

and passing to the limit we arrive at

$$\lim_{n\to\infty} \int_0^n G_n(n/\alpha, s)\, g(s, y(s))\, ds = \frac{\alpha - 1}{\alpha}\, y_0 \in [r_0, 0).$$

On the other hand, by $(f1)$ there exist $\bar{\tau} \geq \tau_1$ and $c < 0$ such that

$$f(t(\tau), y(\tau)) < c \quad \text{for all } \tau > \bar{\tau}.$$

Moreover, denoting by

$$\beta = \frac{1-\alpha}{\alpha} \int_0^{\bar{\tau}} s\, g(s, y(s))\, ds,$$

and using (2.2), we deduce that the following inequalities hold for all $n \geq \alpha\, \bar{\tau}$

$$\begin{aligned}
\int_0^n G_n(n/\alpha, s)\, g(s, y(s))\, ds &= \int_0^{\bar{\tau}} G_n(n/\alpha, s)\, g(s, y(s))\, ds \\
&\quad + \int_{\bar{\tau}}^n G_n(n/\alpha, s)\, g(s, y(s))\, ds \\
&= \beta + \int_{\bar{\tau}}^n G_n(n/\alpha, s)\, p(t(s)) f(t(s), y(s))\, ds \\
&\geq \beta + c \int_{\bar{\tau}}^n G_n(n/\alpha, s)\, p(t(s))\, ds \\
&\geq \beta + c \int_{n/\alpha}^n G_n(n/\alpha, s)\, p(t(s))\, ds \\
&= \beta + \frac{c}{\alpha} \int_{n/\alpha}^n (s - n)\, p(t(s))\, ds.
\end{aligned}$$

Now, from this inequality and condition $(p1)$, we deduce that

$$\lim_{n\to\infty} \int_0^n G_n(n/\alpha, s)\, g(s, y(s))\, ds = +\infty,$$

and we attain a contradiction.

So, $\lim_{\tau\to\infty} y(\tau) = 0$ and the proof is finished. ■

If, instead of condition $(p1)$ we assume

(p2) Suppose that there is $t_1 \in (0,T)$ for which function p is nonincreasing in (t_1,T) and there is $\alpha > 1$ such that
$$\lim_{n\to\infty} n^2\, p(t(n/\alpha)) = +\infty, \tag{2.3}$$

we deduce, as a straightforward consequence of theorem 2.2, the following result.

Corollary 2.3 *If assumptions (p0), (p2), (f0) and (f1) hold then problem (1.3) has a nontrivial odd solution (with respect to $t = T$) which moreover satisfy $r_0 \le u(t) \le 0$ for all $t \in (0,\pi)$ and $u(t) < \gamma(t) < 0$ for all $t \in (\bar{t},T)$, for some $\bar{t} \in (t_0,T)$.*

Proof. It is clear that we only need to verify that condition $(p1)$ is fulfilled. To this end, notice that

$$\begin{aligned}
\int_{n/\alpha}^{n} (n-s)\, p(t(s))\, ds &= \int_{n/\alpha}^{n} \left(\int_{s}^{n} p(t(s))\, dr \right) ds \\
&= \int_{n/\alpha}^{n} \left(\int_{n/\alpha}^{r} p(t(s))\, ds \right) dr \\
&= \int_{n/\alpha}^{n} \left(\int_{t(n/\alpha)}^{t(r)} d\tau \right) dr \\
&= \int_{n/\alpha}^{n} (t(r) - t(n/\alpha))\, dr.
\end{aligned}$$

Now, the mean value theorem imply that there is $\tau_n \in [n/\alpha, r]$ such that

$$t(r) - t(n/\alpha) = t'(\tau_n)\,(r - n/\alpha) = p(t(\tau_n))\,(r - n/\alpha).$$

Thus, for n large enough we know that $t(n/\alpha) \ge t_1$ and from (p2) it follows

$$t(r) - t(n/\alpha) \ge p(t(n/\alpha))\,(r - n/\alpha),$$

and thus

$$\int_{n/\alpha}^{n} (n-s)\, p(t(s))\, ds \ge p(t(n/\alpha)) \int_{n/\alpha}^{n} (r - n/\alpha)\, dr = \frac{(\alpha-1)^2}{2\,\alpha^2}\, p(t(n/\alpha))\, n^2.$$

So, condition $(p2)$ implies that condition $(p1)$ holds and the results of Theorem 2.2 are valid. ∎

Example 2.4 *Let us consider the following problem*

$$\begin{cases} (p(t)x')' - q(t)x = A\sin\left(\frac{\pi t}{T}\right), & t \in [0, 2T], \\ x(0) = x(2T), \quad x'(0) = x'(2T). \end{cases}$$

If we assume that p *satisfies* $(p0)$ *and (p1),* $A > 0$ *and moreover*

$(q0)$ $q(T-t) = q(T+t)$ *for all* $t \in [0, T]$,

$(q1)$ $q(t) \geq k > 0$ *for all* $t \in [0, 2T]$,

$(q2)$ q *is nondecreasing on some interval* $[t_2, T]$ *with* $t_2 \in (0, T)$,

then a simple computation shows that $(f0)$ *and* $(f1)$ *hold, with* $t_0 = \max\{T/2, t_2\}$, $r_0 = -A/k < 0$ *and* $\gamma(t) = \dfrac{-A\sin\left(\frac{\pi t}{T}\right)}{q(t)}$*. Thus the existence of a nontrivial odd solution follows from Theorem 2.2.*

Remark 2.5 *Notice that if we are looking for solutions of the Dirichlet problem, instead of periodic ones, then the symmetric conditions (p0) and (f0) are no longer needed.*

For instance, if $p : [0, T] \to \mathbb{R}$ *is continuous,* $p(t) > 0$ *for all* $t \in [0, T)$, $\int_0^T \frac{1}{p(s)} ds = +\infty$ *and conditions (p1), (q1) and (q2) are fulfilled, then problem*

$$\begin{cases} (p(t)x')' - q(t)x = A\sin^2\left(\frac{j\pi t}{T}\right), & t \in [0, T], \\ x(0) = x(T) = 0, \end{cases}$$

has at least a solution $-A/k \leq x(t) \leq 0$ *on* $(0, T)$*, for all* $j \in \mathbb{N}$ *and* $A > 0$*. Moreover*

$$x(t) < -\frac{A\sin^2\left(\frac{j\pi t}{T}\right)}{q(t)} < 0,$$

for all $t \geq \bar{t} \geq \max\{t_2, (2j-1)T/(2j)\}$.

3 An application to problem (1.2)

In this section we will apply the main result given in previous section to problem (1.2) in the singular case $e = 1$. We recall that by a solution of problem (1.2) we mean a function $x \in C([0, 2\pi]) \bigcap C^1([0, \pi) \cup (\pi, 2\pi])$ with $(1 + \cos(t))^2 x' \in C^1([0, 2\pi])$ that satisfies the differential equation and the boundary conditions. The existence result is the following.

Theorem 3.1 *If* $e = 1$ *and* $\lambda \leq -4$ *then problem (1.2) has a nontrivial odd solution (with respect to* $t = \pi$*) which moreover satisfies* $-\frac{\pi}{2} \leq u(t) \leq 0$ *for all* $t \in [0, 2\pi]$ *and* $u(t) < \arcsin(4\sin t/\lambda) < 0$ *for some* $\bar{t} \in (\pi, 2\pi)$.

Proof. As we have already noticed, problem (1.2) with $e = 1$, is a particular case of problem (1.3) with $T = \pi$, $p(t) = (1 + \cos(t))^2$ and

$$f(t, x) = 4(1 + \cos(t))\sin(t) - \lambda(1 + \cos(t))\sin(x).$$

In consequence, conditions $(p0)$ and $(f0)$ are clearly satisfied. On the other hand, it is not difficult to verify that, if $\lambda \leq -4$, condition $(f1)$ holds for $r_0 = -\pi/2$, $\gamma(t) = \arcsin\left((4 \sin t)/\lambda\right)$ and $t_0 = \pi/2$. In the sequel, we shall verify condition $(p2)$ to concluding the result as a consequence of Corollary 2.3.

It is clear that function p is nonincreasing on $(0, \pi)$. Moreover, some computations with Mathematica show us that for all $t \in [0, \pi)$

$$\tau(t) = \int_0^t \frac{1}{(1 + \cos(s))^2} ds = \frac{1}{12}\left(3\sin\left(\frac{t}{2}\right) + \sin\left(\frac{3t}{2}\right)\right)\left(\sec\left(\frac{t}{2}\right)\right)^3,$$

and so its inverse is given for all $\tau \in [0, \infty)$ by

$$t(\tau) = 2\,\mathrm{arcsec}\left(\sqrt{\sqrt[3]{18\tau^2 + 6\sqrt{9\tau^4 + \tau^2} + 1} + \frac{1}{\sqrt[3]{18\tau^2 + 6\sqrt{9\tau^4 + \tau^2} + 1}} - 1}\right).$$

Then for all $s \in [0, \infty)$ we have

$$p(t(s)) = \frac{4}{\left(\sqrt[3]{18s^2 + 6\sqrt{9s^4 + s^2} + 1} + \frac{1}{\sqrt[3]{18s^2+6\sqrt{9s^4+s^2}+1}} - 1\right)^2}.$$

Tacking $\alpha = 2$, we obtain that

$$n^2\, p(t(n/2)) = \frac{4\, n^2}{\left(\sqrt[3]{\frac{9n^2}{2} + \frac{3}{2}\sqrt{n^2\,(9n^2 + 4)} + 1} + \frac{1}{\sqrt[3]{\frac{9n^2}{2}+\frac{3}{2}\sqrt{n^2(9n^2+4)}+1}} - 1\right)^2},$$

and therefore condition (2.3) is satisfied.

■

References

[1] P. Amster and A. Deboli, A Neumann boundary-value problem on an unbounded interval, *Electron. J. Differential Equations* (2008), 90, 5 pp.

[2] R. P. Agarwal and D. O'Regan, *Infinite interval problems for differential, difference and integral equations*, Kluwer Academic Publishers, Dordrecht, (2001).

[3] J. Andres, G. Gabor and L. Górniewicz, Boundary value problems on infinite intervals, *Trans. Amer. Math. Soc.* **351** (1999), 4861–4903.

[4] V. V. Beletskii, On the oscillations of a satellite, *Iskusst. Sputn. Zemli* **3** (1959), 1–3.

[5] V. V. Beletskii, *The motion of an artificial satellite about its centre of mass*, Nauka, Moscow, 1965.

[6] V. V. Beletskii, *Essays on the motion of celestial bodies*, Birkhäuser, 2001.

[7] M. Benchohra, An existence theorem on unbounded intervals for a class of second order functional differential inclusions in Banach spaces, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **9** (2002), 541–549.

[8] P. A. Binding and H. Volkmer, Oscillation theory for Sturm-Liouville problems with indefinite coefficients, *Proc. Roy. Soc. Edinburgh Sect. A* **131** (2001), 989–1002.

[9] A. D. Bruno and V. P. Varin, The limit problems for the equation of oscillations of a satellite, *Celestial Mech. Dynam. Astronom.* **67** (1997), 1–40.

[10] A. D. Bruno and V. P. Varin, Singularities of oscillations of a satellite of highly eccentric elliptic orbits, Proceedings of the Second World Congress of Nonlinear Analysts, *Nonlinear Anal.* **30** (1997), 2541–2546.

[11] C. De Coster and P. Habets, *Two-point boundary value problems: lower and upper solutions*, Mathematics in Science and Engineering, 205 Elsevier B. V., Amsterdam, (2006).

[12] M. Duhoux, Nonlinear singular Sturm-Liouville problems, *Nonlinear Anal.* **38** (1999), 897–918.

[13] M. Duhoux, Strongly singular Sturm-Liouville problems, *Math. Nachr.* **225** (2001), 19–38.

[14] D. D. Hai, Note on a differential equation describing the periodic motion of a satellite in its elliptical orbit, *Nonlinear Anal.* **12** (1988), no. 12, 1337–1338.

[15] D. D. Hai, On a second order periodic boundary value problem, *Differential Integral Equations* **1** (1988), no. 3, 377–379.

[16] D. D. Hai, Multiple solutions for a nonlinear second order differential equation, *Ann. Polon. Math.* **52** (1990), no. 2, 161–164

[17] I. D. Kill, On periodic solutions of a certain nonlinear equation, *J. Appl. Math. Mech.* **27** (1964), 1699-1704.

[18] J. Mawhin, On a differential equation for the periodic motions of a satellite around its center of mass, Asymptotic methods in mathematical physics, 150–157, 302, “Naukova Dumka”, Kiev, 1988.

[19] D. Núñez and P. J. Torres, Stable odd solutions of some periodic equations modeling satellite motion, *J. Math. Anal. Appl.* **279** (2003), no. 2, 700–709.

[20] W. V. Petryshyn and Z. S. Yu, On the solvability of an equation describing the periodic motions of a satellite in its elliptic orbit, *Nonlinear Anal.* **9** (1985), no. 9, 969–975.

[21] J. D. Šlapak, The periodic solutions of second order nonlinear equations that are not solved with respect to the highest derivative, *Ukrain. Mat. Ž.* **26** (1974), 850–854.

[22] A. P. Torzhevskii, Periodic solutions of the equation for two-dimensional vibrations of a satellite with elliptical orbit, *Kosmich. Issled.* **2** (1964), 667–678.

[23] B. Q. Yan, D. O’Regan and R. P. Agarwal, Positive solutions to singular boundary value problems with sign changing nonlinearities on the half-line via upper and lower solutions, *Acta Math. Sin. (Engl. Ser.)* **23** (2007), 1447–1456.