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ON COMPARISON PRINCIPLES FOR THE PERIODIC HILL'S EQUATION

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Abstract

In this work we make an exhaustive study of the properties of the Green's function related to the periodic boundary value problem

$$L_a\, x \equiv x'' + a(t)\; x = 0, \quad x(0) = x(T), \quad x'(0) = x'(T),$$

with a sign-changing potential $a(t)$.

Moreover, we obtain new explicit criteria that ensures the maximum or antimaximum principle holds for this equation. The given criteria complement previous results in the literature.

1. *Introduction*

The topic of maximum and antimaximum principles related to the Hill's operator

$$L_a u(t) \equiv u''(t) + a(t)\, u(t),\; t \in [0,T] \equiv I,$$

with

$$a : \mathbb{R} \to \mathbb{R}, \quad a \in L^{\alpha}(I),\, \alpha \geq 1 \quad \text{and} \quad a(t+T) = a(t) \quad \text{a. e. } \; t \in \mathbb{R}, \tag{1.1}$$

has been widely studied in the literature [**1**, **5**, **10**, **18**, **22**, **28**, **29**]. These comparison principles are fundamental tools when we consider nonlinear boundary value problems and apply, among others, monotone iterative techniques [**11**, **13**], lower and upper solutions method [**3**, **7**], fixed points theorems [**11**, **22**] or stability theory [**29**].

To fix ideas, we will denote by $AC^1(I)$ the space of the absolutely continuous functions on I whose first derivative is also an absolutely continuous function on I and let X be the Banach space

$$X = \{u \in AC^1(I),\, u(0) = u(T),\; u'(0) = u'(T)\}.$$

We say that L_a (with periodic boundary conditions) admits the maximum principle (MP) if and only if

$$u \in X,\, L_a u \geq 0 \,\text{on}\, I \Longrightarrow u \equiv 0 \quad \text{or} \quad u < 0 \quad \text{on } I,$$

and L_a (with periodic boundary conditions) admits the antimaximum principle (AMP) if and only if

$$u \in X,\, L_a u \geq 0 \,\text{on}\, I \Longrightarrow u \equiv 0 \quad \text{or} \quad u > 0 \quad \text{on } I.$$

We say that *operator L_a is nonresonant in X* if and only if the homogeneous problem

$$L_a\, u(t) = 0 \quad \text{a. e. } \; t \in I,\, u(0) = u(T),\; u'(0) = u'(T), \tag{1.2}$$

has only the trivial solution.

Of course, if L_a satisfies MP or AMP then it is nonresonant. Also it is well known that when L_a is nonresonant then for all $\sigma \in L^1(I)$ the problem

$$L_a u(t) = \sigma(t), \quad \text{a. e. } \; t \in I, \qquad u(0) = u(T),\; u'(0) = u'(T), \tag{1.3}$$

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has a unique solution $u \in AC^1(I)$ and there is a unique continuous function $G_a : I \times I \to \mathbb{R}$, such that

$$u(t) = \int_0^T G_a(t,s)\,\sigma(s)\,ds, \qquad \forall\, t \in I, \tag{1.4}$$

$G_a(t,s)$ being the so-called Green's function related to the operator L_a in X.

Moreover one can easily verify that operator L_a is self-adjoint on X and that this property is equivalent to the fact that the Green's function is symmetrical with respect to the diagonal of its square of definition, that is,

$$G_a(t,s) = G_a(s,t), \qquad \forall\, (t,s) \in I \times I. \tag{1.5}$$

It is also known that $G_a \geq 0$ on $I \times I$ is equivalent to the AMP and $G_a \leq 0$ on $I \times I$ is equivalent to the MP (see [**28**, Theorem 4.1]). So every assertion about the constant sign of the Green's function G_a is also an assertion about a comparison principle for L_a.

The paper is organized as follows: in Section 2 we obtain some useful properties concerning the sign of the periodic Green's function. In Section 3 we review the known comparison principles for operator L_a with periodic boundary conditions. Although optimal conditions for such comparison principles were obtained in [**28**] in terms of eigenvalues, Green's functions or rotation numbers, new explicit criteria are still welcome. To this end, we obtain a characterization for the constant sign of the Green's function by means of the oscillatory properties of operator L_a. In Section 4 we obtain some new explicit criteria for MP and AMP, focusing our attention mainly into the cases less studied up to now. Finally in Section 5 we present some examples showing the applicability of our results and comparing them with those available in the related literature.

Some notation is needed throughout the paper: we will denote by $h \succ 0$ a function $h \in L^\alpha(0,T)$ such that $h(t) \geq 0$ for a. e. $t \in I$ and $h \not\equiv 0$ on I. For $1 \leq \alpha \leq \infty$ we denote by α^* its conjugate, that is, $\dfrac{1}{\alpha} + \dfrac{1}{\alpha^*} = 1$, (with $\alpha = 1$ and $\alpha^* = \infty$ and vice-versa). Finally, denoting $H_0^1(0,T)$ as the usual Sobolev space of the $AC^1(I)$ functions that satisfy the Dirichlet conditions, we define $K(\alpha,T)$ as the best Sobolev constant in the inequality

$$C\|u\|_\alpha^2 \leq \|u'\|_2^2 \quad \text{for all } u \in H_0^1(0,T),$$

given explicitly by, see [**20**],

$$K(\alpha,T) = \begin{cases} \dfrac{2\pi}{\alpha T^{1+2/\alpha}} \left(\dfrac{2}{2+\alpha}\right)^{1-2/\alpha} \left(\dfrac{\Gamma(1/\alpha)}{\Gamma(1/2+1/\alpha)}\right)^2, & \text{if } 1 \leq \alpha < \infty, \\[2ex] \dfrac{4}{T}, & \text{if } \alpha = \infty. \end{cases}$$

We note that in all the paper we assume that potential a satisfies condition (1.1). Moreover we let G_a denote the $T-$periodic extension in the two variables of the Green's function.

2. *Properties of the Green's functions*

This section is devoted to the study of some useful properties of the Green's function related to operator L_a coupled with the periodic boundary conditions, that is, defined in the space X.

Let now $s \in I$ be given. From the definition of the Green's function, using the $T-$periodicity of the function a, and denoting by $x_s(\cdot)$ the function $G_a(\cdot,s)$, it is not difficult to verify that $x_s \in W^{2,1}(I_s^k)$, where $I_s^k = (s + k\,T, s + (k+1)\,T)$, and it satisfies the equation

$$x_s''(t) + a(t)\,x_s(t) = 0, \text{ a. e. } t \in I_s^k, \quad k \in \mathbb{Z}. \tag{2.1}$$

LEMMA 2.1. *The four following equalities are equivalent:*
(i) $G_a(T,0) = 0$.
(ii) $G_a(0,T) = 0$.
(iii) $G_a(0,0) = 0$.
(iv) $G_a(T,T) = 0$.

Proof. Suppose that $G_a(T,0) = 0$ then, by (1.5) we deduce that $G_a(0,T) = 0$. The rest of the proof follows by the periodicity of the Green's function. □

Now, we obtain the points in which a constant sign Green's function can vanish.

LEMMA 2.2. *Suppose that the Green's function G_a does not change sign on $I \times I$ and G_a vanishes at some point $(t_0, s_0) \in I \times I$, then $t_0 = s_0$, $(t_0, s_0) = (0,T)$ or $(t_0, s_0) = (T, 0)$.*

Proof. Suppose, on the contrary, that $G_a(t_0, s_0) = 0$ for some $(t_0, s_0) \in (0,T) \times (0,T)$ such that $t_0 \neq s_0$. Since $G_a(t_0, s_0) = G_a(s_0, t_0)$, we may assume $t_0 > s_0$. From equation (2.1), we know that function

$$x(t) \equiv G_a(t, s_0), \qquad t \in \mathbb{R},$$

solves the equation

$$x''(t) + a(t)\,x(t) = 0, \qquad \text{a. e. } t \in (s_0, s_0 + T), \quad x(t_0) = x'(t_0) = 0.$$

So, $G_a(t, s_0) = 0$ for all $t \in (s_0, s_0 + T)$. But this contradicts the fact that

$$\frac{\partial G_a}{\partial t}(t^+, t) - \frac{\partial G_a}{\partial t}(t^-, t) = 1, \quad \text{for all } t \in \mathbb{R}. \tag{2.2}$$

□

REMARK 2.1. *If we consider the constant potential $a(t) \equiv \left(\frac{\pi}{T}\right)^2$, we know that the Green's function is strictly positive on $I \times I$ except at the diagonal and at the points $(0,T)$ and $(T,0)$. In consequence the previous result is optimal.*

Therefore, we arrive at the following conclusion.

COROLLARY 2.1. *If the Green's function G_a does not change sign on $I \times I$, then, for all $t \in I$ the functions $G_a(t,\cdot)$ and $G_a(\cdot, s)$ vanish on I in, at most, two points. Even more, in such a case we have:*
(i) *If $s_0 = 0$ or $s_0 = T$, then $G_a(\cdot, s_0)$ can only vanish at $t = 0$ and $t = T$.*
(ii) *If $t_0 = 0$ or $t_0 = T$, then then $G_a(t_0, \cdot)$ can only vanish at $s = 0$ and $s = T$.*
(iii) *If $s_0 \in (0,T)$, then $G_a(\cdot, s_0)$ can only vanish at $t = s_0$.*
(iv) *If $t_0 \in (0,T)$, then $G_a(t_0, \cdot)$ can only vanish at $s = t_0$.*

Now, we will obtain the relation between the sign of the Green's function G_a and the comparison principles for L_a. Such result is known (see for instance [**28**, Theorem 4.1]), but we present a proof for the sake of completeness.

LEMMA 2.3. *The following claims are equivalent:*
(1) *$G_a(t,s) \geq 0$ (≤ 0) on $I \times I$.*
(2) *If $x \in X$ and $L_a\,x \succ 0$ on I then $x > 0$ (< 0) on I.*

Proof. First observe that inequality $L_a\,x \succ 0$ on I is equivalent to the existence of some $\sigma \in L^1(I)$ such that $\sigma \succ 0$ on I, for which equation (1.3) is fulfilled. If the Green's function does not change sign, we deduce the constant sign of x on I as a direct consequence of (1.4) and Corollary 2.1.

Reciprocally, assume *(2)* and suppose that G_a changes sign on $I \times I$. Arguing as in [**3**, Theorem 3.1], one can find $t_0 \in I$ and $x_1,\, x_2 \in X$ such that $L_a\, x_1 \succ 0$, $L_a\, x_2 \succ 0$ on I and $x_1(t_0)\, x_2(t_0) < 0$, which is a contradiction with *(2)*. □

As a consequence of the two previous results, we deduce the following property for nonpositive Green's functions.

LEMMA 2.4. *If $G_a(t,s) \le 0$ on $I \times I$ then $G_a(t,s) < 0$ on $I \times I$.*

Proof. From Lemmas 2.1 and 2.2 we have two possibilities:

(i) *There exists $t_0 \in (0,T)$ such that $G_a(t_0,t_0) = 0$.*
Since G_a is nonpositive, we know, from (2.2), that

$$\frac{\partial G_a}{\partial t}(t_0^+, t_0) = \frac{\partial G_a}{\partial t}(t_0^-, t_0) + 1 \ge 1,$$

which implies that G_a is positive on a right-neighborhood of t_0, and we attain a contradiction.

(ii) $G_a(0,0) = 0$.
From (2.1), we have that

$$x_0''(t) + a(t)\, x_0(t) = 0, \text{ a. e. } t \in I_0^k, \quad k \in \mathbb{Z},$$

and from (2.2) we obtain

$$x_0'(0^+) = x_0'(0^-) + 1 = x_0'(T^-) + 1.$$

On the other hand, Lemma 2.1 implies that $x_0(0) = x_0(T) = 0$ and the nonpositiveness of x give us

$$x_0'(0^+) \le 0 \le x_0'(T^-) < x_0'(0^+),$$

which is a contradiction.

□

From the definition of x_s given in (2.1) and reasoning as in the previous lemmas, we deduce the following result.

LEMMA 2.5. *Suppose that the Green's function G_a does not change sign on $I \times I$. Then it is nonnegative on $I \times I$ and vanishes at some point $(t_0,t_0) \in I \times I$ if and only if the equation*

$$x''(t) + a(t)\, x(t) = 0, \quad t \in (t_0, t_0 + T), \quad x(t_0) = x(T + t_0) = 0$$

has a non trivial constant sign solution.

Defining now the function

$$a_s(t) \equiv a(t+s), \quad s,\ t \in \mathbb{R}.$$

we arrive at the following result.

LEMMA 2.6. *For all $t,\ s \in \mathbb{R}$ we have* $\quad G_a(t,s) = G_{a_s}(t-s, 0)$.

Proof. From the periodicity of G_a and condition (2.2), we have that $x_s(\cdot) := G_a(\cdot, s)$ is the unique solution of the equation

$$x_s''(t) + a(t)\, x_s(t) = 0, \text{ a. e. } t \in (s, s+T), \quad x_s(s) = x_s(s+T),\ x_s'(s^+) = x_s'((s+T)^-) + 1.$$

On the other hand, $y_s(t) := x_s(t+s)$ is the unique solution of the equation

$$y_s''(t) + a(t+s)\, y_s(t) = 0, \text{ a. e. } t \in (0, T), \quad y_s(0) = y_s(T),\ y_s'(0^+) = y_s'(T^-) + 1.$$

As consequence, $x_s(s+t) = G_{a_s}(t, 0)$ or, which is the same, $G_a(t,s) \equiv x_s(t) = G_{a_s}(t-s, 0)$. □

REMARK 2.2. *We notice that the previous property extends to a non constant potential $a(t)$ the expression obtained in [**3**, Lemma 2.1] for constant ones.*

Lemma 2.6 allows us to rewrite Lemma 2.5 as follows.

COROLLARY 2.2. *Suppose that the Green's function G_a does not change sign on $I \times I$. Then it vanishes at some point $(t_0, t_0) \in I \times I$ if and only if the equation*

$$x''(t) + a_{t_0}(t)\, x(t) = 0, \quad t \in I, \qquad x(0) = x(T) = 0,$$

has a non trivial constant sign solution.

Moreover, we deduce the following result.

LEMMA 2.7. *Let $b(t) \equiv a(T-t)$ for all $t \in \mathbb{R}$. Then the following equality holds:*

$$G_a(t,s) = G_b(T-t, T-s) \quad \textit{for all} \quad t,\ s \in \mathbb{R}.$$

Proof. We know that $x(t) := G_{a_s}(t, 0)$ is the unique solution of equation

$$x''(t) + a(t+s)\, x(t) = 0, \text{ a. e. } t \in (0, T), \quad x(0) = x(T),\ x'(0^+) = x'(T^-) + 1.$$

So, $y(t) = x(T-t)$ is the unique solution of equation

$$y''(t) + b(t+s)\, y(t) = 0, \text{ a. e. } t \in (0, T), \quad y(0) = y(T),\ y'(0^+) = y'(T^-) + 1,$$

that is, $y(t) := G_{a_s}(T-t, 0) = G_{b_s}(t, 0)$.

Now, from Lemma 2.6 and the properties of the Green's function, we deduce that

$$\begin{aligned} G_a(t,s) &= G_{a_s}(t-s, 0) = G_{b_s}(T-t+s, 0) \\ &= G_b(T-t, -s) = G_b(T-t, T-s). \end{aligned}$$

□

As a consequence of the two previous lemmas we can conclude the following corollary.

COROLLARY 2.3. *Let a_s and b_r be defined as in the two previous lemmas for some s, $r \in I$, then the functions G_{a_s} and G_{b_r} take exactly the same values (at different points) on $I \times I$.*

To finish this section, we obtain the following comparison results for the Green's functions related to different potentials.

LEMMA 2.8. *Let a_1, $a_2 \in L^\alpha(I)$ be such that the corresponding Green's functions of the periodic problem G_{a_1} and G_{a_2} do not change sign on $I \times I$ and $G_{a_1}(t,s)\, G_{a_2}(t,s) \geq 0$ for all $(t,s) \in I \times I$. If $a_1 \succ a_2$ on I then $G_{a_1}(t,s) < G_{a_2}(t,s)$ for all $(t,s) \in I \times I$.*

Proof. Let $s \in I$ be given. Denote by $x_s(\cdot) = G_{a_1}(\cdot, s)$ and $y_s(\cdot) = G_{a_2}(\cdot, s)$. From the definition of the Green's functions, we know that

$$x_s''(t) + a_1(t)\, x_s(t) = y_s''(t) + a_2(t)\, y_s(t) = 0, \text{ a. e. } t \in (s, s+T),$$

and

$$x_s(s) = x_s(s{+}T), \quad y_s(s) = y_s(s{+}T), \quad x_s'(s^+) = x_s((s{+}T)^-){+}1, \quad y_s'(s^+) = y_s'((s{+}T)^-){+}1.$$

Now, for all $t \in I$, we define the functions $\bar{x}_s(t) := x_s(t+s)$ and $\bar{y}_s(t) := y_s(t+s)$. Therefore

$$\bar{x}_s - \bar{y}_s \in X.$$

Suppose now that the Green's functions are nonnegative (the other case is analogous). In consequence, from Corollary 2.1 it follows that $\bar{x}_s > 0$ and $\bar{y}_s > 0$ for all $t \in (0, T)$.

Let $\epsilon(t) = a_1(t) - a_2(t) \succ 0$ on I. We have

$$(\bar{y}_s - \bar{x}_s)''(t) + a_1(t+s)\, (\bar{y}_s - \bar{x}_s)(t) = \epsilon(t+s)\, \bar{y}_s(t) \succ 0, \text{ a. e. } t \in (0, T),$$

and we deduce, from Lemma 2.3, that $\bar{y}_s > \bar{x}_s$ on I, or, which is the same, $G_{a_1}(t, s) < G_{a_2}(t, s)$ for all $(t, s) \in I \times I$. □

3. *A survey on MP and AMP for operator L_a*

In this section we present the state of the art of comparison principles for the periodic Hill's equation. In the first subsection we show a characterization due to M. Zhang by using the corresponding eigenvalues of the related homogeneous equation. Such characterization allows us to describe the constant sign Green's functions from the oscillation properties of operator L_a.

3.1. *Optimal conditions for MP and AMP*

Let $\overline{\lambda}_0(a)$ be the smallest eigenvalue of the periodic equation

$$u''(t) + (a(t) + \lambda)\, u(t) = 0, \quad \text{a. e. } t \in I, \qquad u(0) = u(T),\ u'(0) = u'(T),$$

and let $\underline{\lambda}_1(a)$ be the smallest eigenvalue of the anti-periodic equation

$$u''(t) + (a(t) + \lambda)\, u(t) = 0, \quad \text{a. e. } t \in I, \qquad u(0) = -u(T),\ u'(0) = -u'(T).$$

In [**28**], M. Zhang obtained the following result (in the paper for $T = 1$).

Lemma 3.1. *[**28**, Theorem 1.1] Suppose that $a \in L^1(I)$, then:*
(i) *L_a admits MP if and only if $\overline{\lambda}_0(a) > 0$.*
(ii) *L_a admits AMP if and only if $\overline{\lambda}_0(a) < 0 \le \underline{\lambda}_1(a)$.*

By introducing the parameterized potentials $\lambda + a$ the previous result can be rewritten as follows.

Lemma 3.2. *[**28**, Theorem 1.2] Suppose that $a \in L^1(I)$, then:*
(i) *$L_{\lambda+a}$ admits MP if and only if $\lambda < \overline{\lambda}_0(a)$.*
(ii) *$L_{\lambda+a}$ admits AMP if and only if $\overline{\lambda}_0(a) < \lambda \le \underline{\lambda}_1(a)$.*

Remark 3.1. *We point out that in [**28**] are also given optimal conditions for MP and AMP in terms of rotation numbers and the sign of the Green's function. However the characterization by using eigenvalues, as in Lemma 3.1, is more suitable for our purposes.*

The following explicit bounds for the first periodic and anti-periodic eigenvalues are well-known:

LEMMA 3.3. *Suppose that $a \in L^1(I)$, then:*
(i) $\overline{\lambda}_0(a) \leq -\frac{1}{T}\int_0^T a(s)ds$ *and the equality holds if and only if* a *is constant. (See* [**27**]*).*
(ii) $\|a_+\|_\alpha \leq K(2\,\alpha^*, T) \Longrightarrow \underline{\lambda}_1(a) \geq \left(\frac{\pi}{T}\right)^2 \left(1 - \frac{\|a_+\|_\alpha}{K(2\,\alpha^*, T)}\right) \geq 0$. *(See* [**29**]*).*
(iii) $\underline{\lambda}_1(a) = \min\{\lambda_1^D(a_s),\ s \in \mathbb{R}\}$. *Here (see* [**29**]*)* $\lambda_1^D(a_s)$ *is the first eigenvalue of the Dirichlet problem*

$$u''(t) + (a_s(t) + \lambda)\, u(t) = 0, \quad \text{a. e. } t \in I, \quad u(0) = u(T) = 0. \tag{3.1}$$

Now, as a consequence of the previous result and Lemma 2.8, we arrive at the following result.

COROLLARY 3.1. *Let $a_1, a_2 \in L^\alpha(I)$ be such that $a_1 \succ a_2$ on I and assume that the Green's function of G_{a_1} and G_{a_2} have the same constant sign on $I \times I$. Then operator L_a is nonresonant in X and G_a has the same constant sign for all $a \in L^\alpha(I)$ with $a(t) \in [a_2(t), a_1(t)]$ for a. e. $t \in I$.*

Proof. The proof follows for L_a nonresonant in X, since for all $a \in L^\alpha(I)$ and $s \in I$ the function $\lambda_s : L^\alpha(I) \to \mathbb{R}$, that assigns to every potential a_s the first eigenvalue $\lambda_1^D(a_s)$ of the Dirichlet problem (3.1), is decreasing on a_s. Now Lemma 3.3, (*iii*), says us that the same holds for $\underline{\lambda}_1(a)$.

Suppose now that G_{a_1} and G_{a_2} are non negative (the other case is analogous) and there is $a \in L^\alpha(I)$ with $a(t) \in [a_2(t), a_1(t)]$ such that L_a is not invertible in X. Then from the continuous dependence of the Green's function with respect to its potential and the property given above, we deduce that the set

$$\{G_b(t,s),\, (t,s) \in I \times I, \text{with } b \in L^\alpha(I) \text{ and } b(t) \in [a_2(t), a_1(t)]\}$$

is unbounded from below, in contradiction with Lemma 2.8. □

As a consequence of the results showed in this section we arrive at the following characterization of the sign of the Green's function.

THEOREM 3.1. *Let R be the infimum of the distance of two consecutive zeroes of a solution of the equation $L_a\, x = 0$. Then the following assertions hold:*
(i) *G_a changes sign in $I \times I$ if and only if $R < T$.*
(ii) *G_a is non negative and vanishes at some points on $I \times I$ if and only if $R = T$.*
(iii) *G_a has strict constant sign in $I \times I$ if and only if $R > T$.*

Proof. If $R < T$, we have that there is $s \in I$ for which at least one solution of the equation $x''(t) + a_s(t)\, x(t) = 0$, has two zeroes in I. From classical Sturm – Liouville theory, we have that $\lambda_1^D(a_s)$, the first eigenvalue of the Dirichlet problem $L_{a_s}\, x = 0$ on I, $x(0) = x(T) = 0$, is strictly negative. In consequence, Lemma 3.3 (*iii*), implies that $\underline{\lambda}_1(a) < 0$. Now, Lemma 3.1 ensures that G_a changes sign on $I \times I$.

When $R = T$, we conclude, as above, that $\underline{\lambda}_1(a) = 0$. So Lemma 3.1 says us that G_a is non negative on $I \times I$. So, Corollary 2.2 shows that G_a vanishes at some points on $I \times I$.

When $R > T$ we have that G_a has strict constant sign on $I \times I$ (strictly positive or strictly negative) from [**22**, Theorem 2.1]. □

In the next two subsections we present some explicit criteria, in terms of the integral of the potential a (or a_+ and a_-) that ensure the validity of the comparison results.

3.2. *Explicit criteria for AMP*

P. Torres proves the following criteria that ensures the AMP for nonnegative (and not identically zero) potentials:

LEMMA 3.4. *[**22**, Corollary 2.3] Assume that $a \succ 0$ and moreover*

$$\|a\|_\alpha \leq K(2\,\alpha^*, T).$$

Then L_a satisfies the AMP.

The AMP is studied in [**5**] for non constant sign potentials with positive average.

LEMMA 3.5. *[**5**, Theorem 3.2] Assume that $\int_0^T a(t)dt > 0$ and moreover*

$$\|a_+\|_\alpha < K(2\,\alpha^*, T).$$

Then L_a satisfies the AMP.

In [**6**], by studying anti-maximum principles for the quasilinear equation

$$(|u'|^{p-2}u')' + a(t)\,(|u|^{p-2}u) = h(t), \quad u(0) = u(T),\ u'(0) = u'(T),$$

the previous result is extended to the potentials with nonnegative average as follows

LEMMA 3.6. *[**6**, Theorem 3.4 and Remark 3.7] Assume that $\int_0^T a(t)dt \geq 0$, $a \not\equiv 0$, and moreover*

$$\|a_+\|_\alpha \leq K(2\,\alpha^*, T). \tag{3.2}$$

Then L_a satisfies the AMP.

We point that M. Zhang constructs in [**28**] some examples of potentials a for which L_a admits the AMP but inequality (3.2) does not hold.

3.3. *Explicit criteria for MP*

When we refer to the study of MP the following general result for non positive (and not identically zero) potentials was obtained by P. Torres.

LEMMA 3.7. *[**22**, Corollary 2.2] If $a \prec 0$ then L_a satisfies the MP.*

In [**5**] the authors obtain the following result.

LEMMA 3.8. *[**5**, Theorem 4.1] Assume that $a \in L^1(0,T)$ is of the form*

$$a(t) = b'(t) - b^2(t), \quad b(0) = b(T), \quad \int_0^T b(s)ds \neq 0, \tag{3.3}$$

where b is an absolutely continuous function, then L_a satisfies the MP.

REMARK 3.2. *The main difficulty in order to apply Lemma 3.8 is to determine when the potential $a(t)$ is of the form (3.3). It is easy to see that if $a \not\equiv 0$ satisfies (3.3) then*

$\int_0^T a(s)ds < 0$, but the converse is false as the following example shows: let $a \in L^\infty(0,1)$, $a(t) \not\equiv 0$, put $a(t) = \tilde{a}(t) + \bar{a}$, where $\tilde{a}(t)$ has mean value zero and $\bar{a} = \dfrac{1}{T}\displaystyle\int_0^T a(s)ds$ is its mean value, and let us consider the problem:

$$b'(t) - b^2(t) = \tilde{a}(t) + \bar{a}, \quad b(0) = b(1). \tag{3.4}$$

By making the change $u(t) = -b(t)$, problem (3.4) is equivalent to

$$u'(t) + u^2(t) + \tilde{a}(t) = -\bar{a}, \quad u(0) = u(1), \tag{3.5}$$

*and [**17**, Corollary 3.1] implies that there exists $s_0 \in \mathbb{R}$ such that*

(i) *for $-\bar{a} < s_0$ problem (3.5) has no solution,*
(ii) *for $-\bar{a} = s_0$ problem (3.5) has at least one solution,*
(iii) *for $-\bar{a} > s_0$ problem (3.5) has at least two solutions.*

Since problem (3.5) has no solution for $\bar{a} = 0$, as it is easy to see by integrating the equation over $[0,1]$, it follows that $s_0 > 0$. So (i) implies that for mean values satisfying $-s_0 < \bar{a} < 0$ problem (3.5) has no solution, and thus there exist potentials with negative average such that (3.3) has no solution.

Recently, R. Hakl and P. Torres, give the following criterium to ensure the MP.

Lemma 3.9. *[**10**, Corollary 2.5] If $a \in L^1(I)$, $a \not\equiv 0$, and moreover*

$$\int_0^T a_+(s)ds < \frac{4}{T}, \qquad \frac{\int_0^T a_+(s)ds}{1 - \frac{T}{4}\int_0^T a_+(s)ds} \le \int_0^T a_-(s)ds,$$

then L_a satisfies the MP.

Under the assumptions of each one of the previous three results it follows that $\int_0^T a(s)ds < 0$ and in fact this is a necessary condition.

Proposition 3.1 Necessary condition for MP. *If L_a satisfies the MP then $\int_0^T a(s)ds < 0$.*

Proof. The results follows from Lemma 3.1, (1) and Lemma 3.3, (i). □

Corollary 3.2. *Suppose that $\int_0^T a(s)ds \ge 0$. Then the two following assertions hold:*

(i) *If operator L_a is nonresonant on X then it does not satisfy the MP.*
(ii) *$\overline{\lambda}_0(a) \le 0$.*

Proof. The first assertion is just Proposition 3.1. The second one is a direct consequence of Lemma 3.1, (1), coupled with the first part. □

4. *Main results*

This section is devoted to the study of the indefinite potentials for which the MP or the AMP holds and the conditions of the explicit criteria presented in the previous section are not satisfied. In particular, we will pay special attention to the open situations

$$a \not\prec 0 \text{ and } \int_0^T a(t)\, dt < 0$$

or/and

$$\|a_+\|_p > K(2p^*, T).$$

Firstly, we define the functions v_a and w_a as the unique solutions of the following initial value problems:

$$v_a''(t) + a(t)\ v_a(t) = 0, \qquad \text{a. e. } t \in I, \qquad v_a(0) = 0,\ v_a'(0) = 1,$$

and

$$w_a''(t) + a(t)\ w_a(t) = 0, \qquad \text{a. e. } t \in I, \qquad w_a(0) = 1,\ w_a'(0) = 0.$$

Clearly v_a and w_a are a pair of fundamental solutions to the equation $L_a u = 0$. Define now,

$$N_a = \begin{pmatrix} v_a(T) & v_a'(T) - 1 \\ 1 - w_a(T) & -w_a'(T) \end{pmatrix}.$$

In the sequel we use the following characterization of the eigenvalues of problem (1.2), which is essentially [**16**, Theorem 2.1]. For a complete study of such equalities, the reader is referred to [**16**, Chapter II].

Theorem 4.1. *Assume that function a satisfies (1.1), then*
(i) *λ is a periodic eigenvalue if and only if* $\det(N_{a+\lambda}) = 0$.
(ii) *λ is an anti-periodic eigenvalue if and only if* $\det(N_{a+\lambda}) = 4$.
In particular, problem (1.2) is nonresonant if and only if $\det N_a \neq 0$.

Example 4.1. *Suppose that $a(t) \equiv a \in \mathbb{R}$. In this situation, it is not difficult to verify that*

$$v_a(t) = \begin{cases} \frac{\sinh\sqrt{-a}\,t}{\sqrt{-a}}, & \text{if } a < 0 \\ t, & \text{if } a = 0 \\ \frac{\sin\sqrt{a}\,t}{\sqrt{a}}, & \text{if } a > 0, \end{cases} \quad \text{and} \quad w_a(t) = \begin{cases} \cosh\sqrt{-a}\,t, & \text{if } a < 0 \\ 1, & \text{if } a = 0 \\ \cos\sqrt{a}\,t, & \text{if } a > 0. \end{cases}$$

In consequence

$$\det N_a = \begin{cases} 2\,(1 - \cosh\sqrt{-a}\,T), & \text{if } a < 0 \\ 0, & \text{if } a = 0 \\ 2\,(1 - \cos\sqrt{a}\,T), & \text{if } a > 0. \end{cases}$$

So, we deduce the very well known result that this problem is nonresonant if and only if $a \neq (\frac{2\,n\,\pi}{T})^2$, for all $n = 0, 1, \ldots$

Proposition 4.1. *The following assertions hold:*
(i) *If L_a admits MP then* $\det(N_a) < 0$.
(ii) *If L_a admits AMP then* $\det(N_a) > 0$.

Proof. The proof will be made in four steps.
Claim 1. If $a \in L^\infty$ and $a \prec 0$ then $\det(N_a) < 0$.
If $a \prec 0$ it is easy to see that $w_a > 1$ and $v_a' > 1$ on I and so

$$\det(N_a) = 2 - v_a'(T) - w_a(T) < 0.$$

Claim 2. If $a \in L^\infty$ and L_a admits MP then $\det(N_a) < 0$.
Since $a \in L^\infty$, from *Claim 1* it follows that $\det(N_{a+\lambda}) < 0$ for each $\lambda < 0$ such that $a + \lambda \prec 0$.
On the other hand, the mapping $\lambda \in \mathbb{R} \to \det(N_{a+\lambda}) \in \mathbb{R}$ is continuous and vanishes exactly at the periodic eigenvalues (see Theorem 4.1, (1)). Since $\overline{\lambda}_0(a)$ is the smallest periodic eigenvalue, we have that $\det(N_{a+\lambda})$ has constant sign for all $\lambda < \overline{\lambda}_0(a)$. Thus $\det(N_{a+\lambda}) < 0$ for all $\lambda < \overline{\lambda}_0(a)$. From Lemma 3.1, (1), we know that $0 < \overline{\lambda}_0(a)$ and we obtain the desired result, that is, $\det(N_a) < 0$.

Claim 3. If $a \in L^\infty$ and L_a admits AMP then $\det(N_a) > 0$.

By the same argument as in the previous claim we know that $\det(N_{a+\lambda})$ has constant sign for all $\overline{\lambda}_0(a) < \lambda \leq \underline{\lambda}_1(a)$. Moreover, from Lemma 3.1, (2), it is satisfied that $\overline{\lambda}_0(a) < 0 \leq \underline{\lambda}_1(a)$. Now, Theorem 4.1, (2), says us that $\det(N_{a+\underline{\lambda}_1(a)}) = 4 > 0$. In consequence, $\det(N_a) > 0$.

Claim 4. If a satisfies (1.1) then (i) and (ii) hold.

Taking into account that L^∞ is dense in L^α, the result follows from a standard approximation procedure. □

As a consequence of the proof of the previous result, and using the characterization of M. Zhang given in Lemma 3.1, we deduce the following equivalent characterization of the MP and the AMP properties for operator L_a.

THEOREM 4.2. *The following properties hold:*

(i) *L_a satisfies MP if and only if* $\det(N_{a+\lambda}) < 0$ *for all* $\lambda \leq 0$.
(ii) *L_a satisfies AMP if and only if* $\det(N_{a+\lambda}) \leq 4$ *for all* $\lambda \leq 0$ *and* $\det(N_a) > 0$.

Proof. From Theorem 4.1, the eigenvalues of the periodic problem are given as the roots of equation $\det(N_{a+\lambda}) = 0$. In particular, the property $\det(N_{a+\lambda}) < 0$ for all $\lambda \leq 0$ implies that $\overline{\lambda}_0(a) > 0$. But this last assertion is, from Lemma 3.1 (1), equivalent to ensure the MP property for operator L_a.

Assume now that L_a satisfies MP. As we have seen in the proof of the previous result, $\det(N_{a+\lambda}) < 0$ for all $\lambda < \overline{\lambda}_0(a)$. Lemma 3.1 (1), shows us that $\overline{\lambda}_0(a) > 0$, thus $\det(N_{a+\lambda}) < 0$ for all $\lambda \leq 0$, and the first part of the enunciate is proved.

To prove the second one, we use that the roots of equation $\det(N_{a+\lambda}) = 4$ give us the eigenvalues of the anti-periodic problem. From Lemma 3.1 (2), the proof is a direct consequence of the following facts:

(i) $\det(N_{a+\lambda}) \leq 4$ for all $\lambda \leq 0$ if and only if $\underline{\lambda}_1(a) \geq 0$.
(ii) $\det(N_a) > 0$ in and only if $\overline{\lambda}_0(a) < 0$.

The first assertion follows from Theorem 4.1 (2), and the fact showed in the proof of the previous result, that $\det(N_{a+\lambda}) < 0$ for λ small enough. The second one follows from this last property and because of $\overline{\lambda}_0(a)$ is the smallest root of equation $\det(N_a) = 0$. □

We note that to obtain the explicit expression of $\det(N_{a+\lambda})$ is in general not possible for non constant potentials $a(t)$. However there are a lot of very good computer programs, for instance *Maple*, *Mathematica* or *Maxima*, that allow us to get an approximate numerical expression of this formula and its corresponding roots.

Finally, we present a more suitable criteria to ensure the MP and the AMP character of operator L_a. We note that these conditions depend on the integral of the potential $a(t)$.

THEOREM 4.3. *Assume* $\|\tilde{a}_+\|_\alpha \leq K(2\,\alpha^*, T)$. *Then*

(i) *L_a satisfies MP if and only if* $\displaystyle\int_0^T a(s)ds < 0$ *and* $\det(N_a) < 0$.
(ii) *If* $\displaystyle\int_0^T a(s)ds < 0$ *and* $\det(N_a) > 0$ *then L_a satisfies AMP.*
(iii) *If L_a is nonresonant in X and* $\displaystyle 0 \leq \int_0^T a(s)ds \leq \frac{\pi^2}{T}\left(1 - \frac{\|\tilde{a}_+\|_\alpha}{K(2\,\alpha^*, T)}\right)$ *then L_a satisfies AMP.*

Proof. We write $a(t) = \tilde{a}(t) + \lambda$, where $\tilde{a}(t)$ has mean value zero and $\lambda = \dfrac{1}{T}\displaystyle\int_0^T a(s)ds$ is the mean value of a. Since $\|\tilde{a}_+\|_\alpha \le K(2\,\alpha^*, T)$ we have by Lemma 3.3, (ii), that $\underline{\lambda}_1(\tilde{a}) \ge 0$. Moreover, from Lemma 3.6 and Lemma 3.2, (2), we know that $\overline{\lambda}_0(\tilde{a}) < 0$.

To prove the first assertion, suppose that L_a satisfies MP. Then by Propositions 3.1 and 4.1 we obtain that $\int_0^T a(s)ds < 0$ and $\det(N_a) < 0$.

Reciprocally, assume $\int_0^T a(s)ds < 0$ and $\det(N_a) < 0$. Since $\det(N_a) \ne 0$ we have that $L_a = L_{\tilde{a}+\lambda}$ is nonresonant and from $\lambda = \dfrac{1}{T}\displaystyle\int_0^T a(s)ds < 0 \le \underline{\lambda}_1(\tilde{a})$, it follows, from Lemma 3.2, that either L_a admits MP or L_a admits AMP, depending if either $\lambda < \overline{\lambda}_0(\tilde{a})$ or $\overline{\lambda}_0(\tilde{a}) < \lambda < 0$. Finally, $\det(N_a) < 0$ and Proposition 4.1 lead us to conclude that L_a admits MP.

The second part is deduced by repeating the same argument. From the fact that $\lambda = \dfrac{1}{T}\displaystyle\int_0^T a(s)ds < 0 \le \underline{\lambda}_1(\tilde{a})$, we have that either L_a admits MP or L_a admits AMP, so $\det(N_a) > 0$ and Proposition 4.1 imply now that L_a admits AMP.

The last assertion is deduced from Lemma 3.3, (ii), and the following inequalities:

$$\overline{\lambda}_0(\tilde{a}) < 0 \le \lambda = \frac{1}{T}\int_0^T a(s)ds \le \left(\frac{\pi}{T}\right)^2 \left(1 - \frac{\|\tilde{a}_+\|_\alpha}{K(2\,\alpha^*, T)}\right) \le \underline{\lambda}_1(\tilde{a}).$$

□

Remark 4.1.

(i) *The sufficient part in assertion (1) of Theorem 4.3, that is,*

If L_a satisfies MP then $\displaystyle\int_0^T a(s)ds < 0$ and $\det(N_a) < 0$,

is valid for all a satisfying (1.1). In fact it is a direct consequence of Propositions 3.1 and 4.1 in which there are no assumptions on $\|\tilde{a}_+\|_\alpha$.

(ii) *Theorem 4.3 (1) and (2) apply for situations not covered in the related literature. Of course, it remains open to know what happens when the inequality $\|\tilde{a}_+\|_\alpha \le K(2\,\alpha^*, T)$ is not fulfilled.*

(iii) *The assertion (3) in Theorem 4.3 is optimal for constant potentials $a(t) \equiv k$. Moreover, we notice that if $\displaystyle\int_0^T a(s)ds \ge 0$ then $\|\tilde{a}_+\|_\alpha \le \|a_+\|_\alpha$. As a consequence, Theorem 4.3 (3) can cover wider situations than Lemma 3.6.*

5. *Examples*

In this section we present some illustrative examples of our main results.

Example 5.1. *Consider the problem*

$$x''(t) + a_s(t)\, x(t) = h(t), \quad \text{for all } t \in [0,2], \quad x(0) = x(2), \quad x'(0) = x'(2),$$

with

$$a_s(t) = \begin{cases} -1, & \text{if } 0 \le t < 1, \\ s, & \text{if } 1 \le t \le 2, \end{cases}$$

and $s \in \mathbb{R}$.

Of course, for all $s \le 0$, we have that $a_s \prec 0$ and, as a consequence, operator L_{a_s} satisfies the MP.

Now, let $s > 0$. In this case $\int_0^2 a_s(t)dt = s - 1$ and we can check with this family of potentials the available explicit criteria for MP: Lemma 3.7 doesn't apply because a_s is sign-changing. On the other hand, we were not able to verify if a_s is of the form (3.3), so Lemma 3.8 is not useful for us in this case. Finally Lemma 3.9 reads as:

– *If $0 < s < \frac{2}{3}$ then L_{a_s} satisfies the MP.*

Now we compare with Theorem 4.3. We have that

$$\tilde{a}_s(t) = \begin{cases} -\frac{s+1}{2}, & \text{if } 0 \le t < 1, \\ \frac{s+1}{2}, & \text{if } 1 \le t \le 2. \end{cases}$$

Therefore Theorem 4.3 is applicable whenever

$$\|(\tilde{a}_s)_+\|_\alpha = \frac{s+1}{2} \le \max_{\alpha \ge 1} \{K(2\,\alpha^*, 2)\} \approx 2.8125,$$

that is

$$0 < s \le \bar{s} \approx 4.625.$$

After standard computations one can verify that

$$\det(N_{a_s}) = \frac{\left(e^2 - 1\right)(s-1)\sin(\sqrt{s}) - 2\left(1 + e^2\right)\sqrt{s}\cos(\sqrt{s})}{2\,e\,\sqrt{s}} + 2.$$

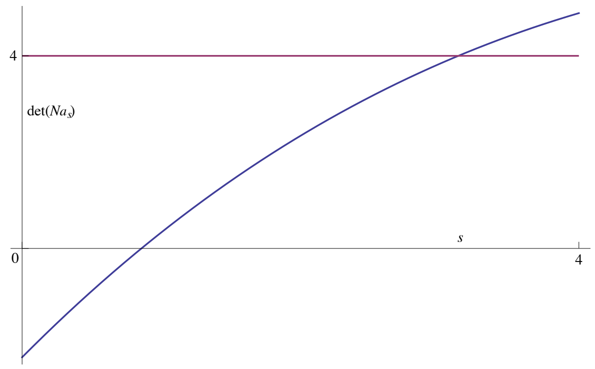

FIGURE 1. *Graphic of* $\det(N_{a_s})$

So, for $s_0 \approx 0.85724$ we have that $\det(N_{a_{s_0}}) = 0$ and $\det(N_{a_s}) < 0$ for $0 < s < s_0$. As consequence, we deduce from Theorem 4.3 the following properties:

– *If $0 < s < s_0$ then L_{a_s} satisfies the MP.*

– *If $s_0 < s < 1$ then L_{a_s} satisfies the AMP.*

Notice that the provided information for the MP case is optimal. Indeed, from Proposition 3.1 we have that a necessary condition for L_{a_s} to satisfy the MP is that $s < 1$. Since we have the AMP when $s_0 < s < 1$ then there is no other possibilities for the MP, that is

– *L_{a_s} satisfies the MP if and only if $0 < s < s_0$.*

On the other hand, we obtain the following estimation for $s \ge 1$ in order to obtain the AMP,

$$1 \le s \le \max_{\alpha \ge 1} \left\{ \frac{\left(1 + \frac{\pi^2}{2} - \frac{\pi^2}{K(2\,\alpha^*,2)}\right)}{\left(1 + \frac{\pi^2}{K(2\,\alpha^*,2)}\right)} \right\} \approx 2.69403.$$

However, this estimation is not the best possible and in fact the bound given in Lemma 3.6 is better than the previous one:

$$1 \le s \le \max_{\alpha \ge 1} \{K(2\,\alpha^*, 2)\} \approx 2.8125.$$

Moreover, since $\det(N_{a_s}) = 4$ *for* $s = s_1 \approx 3.13363$ *and* $\det(N_{a_s}) \in (0,4)$ *for all* $s \in (s_0, s_1)$, *we deduce that* $\underline{\lambda}_1(a_{s_1}) = 0$. *In consequence, from Theorem 4.1 and Lemma 3.2, we have that* $L_{a_{s_1}}$ *satisfies the AMP.*

Now, Corollary 3.1 gives us the following optimal estimate for the AMP:

– L_{a_s} *satisfies the AMP if and only if* $s \in (s_0, s_1]$.

In the previous example the estimation given by Theorem 4.3 to ensure the AMP is worse than the one obtained in Lemma 3.6. Now we present an example where Theorem 4.3 gives a better estimate for AMP.

Example 5.2. *Consider the problem*

$$x''(t) + \frac{\mu}{t\,\log^2 t}\, x(t) = h(t),\ t \in [0, 1/2], \quad x(0) = x(1/2), \quad x'(0) = x'(1/2),$$

with μ *a positive constant. To study the values of the parameter* $\mu > 0$ *for which the MP or the AMP is ensured, we take into account that the potential* $a_\mu(t) := \frac{\mu}{t\,\log^2 t}$ *belongs to* $L^1(0, 1/2)$, *but it does not belong to* $L^\alpha(0, 1/2)$ *for* $\alpha > 1$.

Since $a_\mu(t) > 0$ *for all* $t \in (0, 1/2)$, *we have, from Proposition 3.1, that the corresponding operator* L_{a_μ} *cannot satisfy the MP.*

On the other hand, from the fact that $\|(a_\mu)_+\|_1 = \|a_\mu\|_1 = \mu/\log(2)$ *and* $K(\infty, 1/2) = 8$, *applying Lemma 3.6 or Lemma 3.4, we know that* L_{a_μ} *satisfies the AMP for all* $\mu \in (0, \mu_0]$, *with* $\mu_0 = 8\,\log(2) \approx 5.54518$.

By means of Theorem 4.3, we can improve this estimation as follows: it is obvious that

$$\overline{a}_\mu = 2 \int_0^{1/2} a_\mu(s)\, ds = \frac{2\,\mu}{\log(2)}.$$

So, we deduce that

$$\tilde{a}_\mu(t) := a_\mu(t) - \overline{a}_\mu = \mu \left(\frac{1}{t\,\log^2 t} - \frac{2}{\log(2)} \right)$$

and

$$a_1 \equiv \left\| \left(\frac{1}{t\,\log^2 t} - \frac{2}{\log(2)} \right)_+ \right\|_1 \approx 0.26227.$$

Thus, Theorem 4.3 (3) is rewritten as

$$0 < \mu \le \frac{2\,\pi^2}{\frac{1}{\log(2)} + \frac{\pi^2}{4}\, a_1} \approx 9.44541$$

which is a substantial improvement of the earlier estimate.

In this case it is not possible to get the explicit expressions of functions v_{a_μ} *and* w_{a_μ} *and, as a consequence, of* $\det(N_{a_\mu})$. *However, we can study the related discrete equation and obtain this values with a very small error.*

In particular, it is known (see [**12**]*) that for a given* $n \in \mathbb{N}$ *large enough, the value* $v_{a_\mu}(k/(2\,n)) \approx y(k)$, *for all* $k \in \{1, \dots, n\}$. *Where* $y : \{0, \dots, n\} \to \mathbb{R}$ *is the unique solution of the difference* *equation*

$$y(k+1) - 2\,y(k) + y(k-1) + \frac{\mu}{2\,n\,k\,\log^2(k/(2\,n))}\,y(k) = 0, \quad k \in \{1,\dots,n-1\},$$

coupled with the initial conditions

$$y(0) = 0,\ y(1) = 1/(2\,n).$$

In a similar way, we have an approximation of w_{a_μ}, *by considering the initial conditions* $y(0) = y(1) = 1$.

So by taking $n = 10^6$, *we estimate the first root of the equation* $\det(N_{a_\mu}) = 4$ *by* $\mu = \mu_1 \approx 11.6053$. *Using Corollary 3.1 again, we deduce that the operator* L_{a_μ} *satisfies the AMP if and only if* $\mu \in (0, \mu_1]$.

Finally, we study the AMP for the Mathieu equation.

Example 5.3. *In order to obtain* $2\,\pi$*–periodic positive solutions for the Mathieu equation*

$$x''(t) + (a + b\,\cos t)\;x(t) = h(t),\, t \in [0, 2\pi],\, x(0) = x(2\,\pi),\, x'(0) = x'(2\,\pi), \tag{5.1}$$

with $a \geq 0$, $b \in \mathbb{R}$, $a^2 + b^2 > 0$ *and* $h \succ 0$, *an important tool is the validity of the AMP for operator*

$$Lx := x'' + (a + b\,\cos t)\;x.$$

Notice that, since

$$\int_0^{2\,\pi} (a + b\,\cos s)\,ds = 2\,\pi\,a \geq 0,$$

only the AMP character of this equation has sense.

Since we have that

$$\|\widetilde{(a + b\,\cos t)}_+\|_\alpha = |b|\,\pi^{\frac{1}{2\alpha}} \left(\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)}\right)^{\frac{1}{\alpha}},$$

then Theorem 4.3 (3) means

$$0 \leq |b| \leq (1 - 4\,a) \max_{\alpha \geq 1} \left\{ \frac{K(2\,\alpha^*, 2\,\pi)}{\pi^{\frac{1}{2\alpha}} \left(\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}+1\right)}\right)^{\frac{1}{\alpha}}} \right\}, \tag{5.2}$$

which is shown in the Figure 2.

Moreover, as in the previous examples, Lemma 3.6 gives us an alternative estimation of the admissible values of a *and* b *for which the Green's function is non negative, in this case*

$$\|(a + b\,\cos t)_+\| \leq \max_{\alpha \geq 1} \{K(2\,\alpha^*, 2\,\pi)\}, \tag{5.3}$$

which is shown in the Figure 2 and, as we can observe, in this case is better that the one provided by Theorem 4.3 (3).

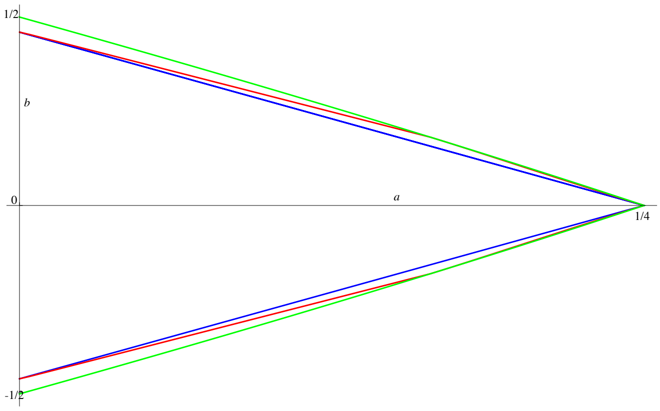

FIGURE 2. *Admissible values of a and b to get the AMP for the Mathieu equation (wider set) together with the graphs of (5.3) (middle) and (5.2) (smaller).*

By using computational methods it is possible to approximate $\det(N_{a+b\,\cos t+\lambda})$ *(see Figure 3) and for instance we have obtained:*

– $\underline{\lambda}_1(0) = 1/4$,
– $\underline{\lambda}_1((\cos t)/4)) \approx 0.17766$,
– $\underline{\lambda}_1(2\,(\cos t)/5) \approx 0.031914$,
– $\underline{\lambda}_1((\cos t)/2) \approx -0.027562$.

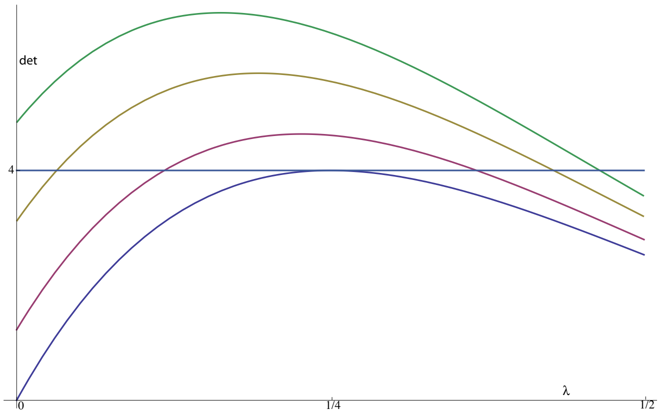

FIGURE 3. *Determinant related to (5.1) for $a = 0$ and $b = 0,\ 1/4,\ 2/5,\ 1/2$ (from below to above).*

REMARK 5.1. *The Brillouin-beam focusing equation*

$$x''(t) + a(1 + \cos(t))\, x(t) = \frac{1}{x(t)},$$

*appears in the study of Electronics, and models the motion of a magnetically focused axially symmetric electron beam under the influence of a Brillouin flow (see [**2**] for details).*

*The existence of positive periodic solutions for this equation has been studied by several authors who give some estimates on the parameter $a > 0$ to ensure such solutions (see for instance [**8**, **9**, **19**, **21**, **22**, **24**, **25**, **26**]). In some of the papers [**21**, **22**] the positiveness of* *the Green's function is fundamental to deduce the existence of solutions. Therefore the results there obtained are automatically valid for the range of parameters showed in Example 5.3.*

*However, in some cases the related Green's function changes sign at some of the values obtained by these authors [**19**]. So the positivity of the Green's function is a sufficient but not a necessary condition to ensure the existence of positive periodic solutions for the Brillouin equation.*

Remark 5.2. *In [**4**, Corollary 2.1] the forced Mathieu-Duffing type equation has been considered*

$$x'' + (a + b\,\cos(t))x - \lambda x^3 = c(t). \tag{5.4}$$

In particular, we prove that if condition (5.3) (with $<$ instead of $\leq$) is satisfied and

$$\min_{t\in[0,2\,\pi]} \int_0^{2\,\pi} G_a(t,s)\,c(s)\,ds > 0$$

then there exists $\lambda_0 > 0$ such that equation (5.4) has at least two positive $2\,\pi$-periodic solutions provided that $0 < \lambda < \lambda_0$.

*Since the proof is obtained from the fact that the Green's function G_a is strictly positive on $[0,2\,\pi]\times[0,2\,\pi]$, we can extend the same result for all the admissible pairs (a,b) shown in the interior of the wider set of Figure 2. Of course the same comment is valid for all the results in [**4**] (or in other papers [**14**, **15**, **21**, **22**, **23**]) based on the positivity of G_a. We do not pursue this line of generalization.*

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