

# A generalized anti-maximum principle for the periodic one dimensional $p$ -Laplacian with sign changing potential

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**Abstract.** It is known that the antimaximum principle holds for the quasilinear periodic problem

$$(|u'|^{p-2}u')' + \mu(t)(|u|^{p-2}u) = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

if  $\mu \geq 0$  in  $[0, T]$  and

$$0 < \|\mu\|_\infty \leq (\pi_p/T)^p, \quad \text{where } \pi_p = 2(p-1)^{1/p} \int_0^1 (1-s^p)^{-1/p} ds,$$

or

$$p = 2 \quad \text{and} \quad 0 < \|\mu\|_\alpha \leq \inf \left\{ \frac{\|u'\|_2^2}{\|u\|_\alpha^2} : u \in W_0^{1,2}[0, T] \setminus \{0\} \right\} \text{ for some } \alpha, 1 \leq \alpha \leq \infty.$$

In this paper we give sharp conditions on the  $L_\alpha$ -norm of the potential  $\mu(t)$  in order to ensure the validity of the antimaximum principle even in case that  $\mu(t)$  can change its sign in  $[0, T]$ .

**Keywords.** Antimaximum principle, periodic problem, Dirichlet problem,  $p$ -Laplacian, singular problem.

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## 1. Introduction

It is well known that the second order periodic boundary value problem

$$(1.1) \quad u'' + \mu u = h(t), \quad u(a) = u(b), \quad u'(a) = u'(b),$$

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where  $-\infty < a < b < \infty$  and  $\mu \in \mathbb{R}$ , satisfies a *maximum principle* (that is,  $h \geq 0$  implies  $u \leq 0$ ) for all  $\mu < 0$  and an *anti-maximum principle* (that is,  $h \geq 0$  implies  $u \geq 0$ ) for all  $0 < \mu \leq (\frac{\pi}{b-a})^2$ . Recall that  $(\frac{\pi}{b-a})^2$  is the first eigenvalue of the corresponding Dirichlet problem (see [5, Theorem 3.1]) and it is an optimal upper bound for  $\mu$  in order to get the anti-maximum principle for the periodic problem (see [2, Lemma 2.5]). We notice that an interesting abstract version of the previous fact has been proved by Campos, Mawhin & Ortega [7] for an operator of the form  $L + \mu I$ , where  $L$  is a linear closed Fredholm operator of index zero and  $I$  is the identity operator.

The introduction of a nonnegative but non constant potential  $\mu \in L_\alpha[a, b]$ , where  $1 \leq \alpha \leq \infty$ , in equation (1.1) makes the problem more difficult to deal with. Recently, Torres & Zhang [19] presented sharp conditions on  $\|\mu\|_\alpha$  ensuring the validity of the anti-maximum principle for the problem

$$u'' + \mu(t)u = h(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

In particular, in the case  $\alpha = \infty$  they recover the classical criterion

$$(1.2) \quad 0 < \|\mu\|_\infty \leq \frac{1}{4}.$$

On the other hand, Cabada, Lomtatidze & Tvrdý [6, Theorem 3.2], dealing with quasilinear operators, have shown that an antimaximum principle holds for

$$(1.3) \quad (\phi_p(u'))' + \mu(t)\phi_p(u) = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

with

$$(1.4) \quad 0 < \|\mu\|_\infty \leq \left(\frac{\pi_p}{T}\right)^p,$$

with  $\pi_p$  defined by

$$(1.5) \quad \pi_p = \frac{2(p-1)^{1/p}}{p} B\left(\frac{1}{p}, \frac{1}{p^*}\right).$$

Here, as usual,  $\phi_p(y) = |y|^{p-2}y$  for  $y \in \mathbb{R}$  stands for the  $p$ -Laplacian with  $1 < p < \infty$ ,  $p^* = \frac{p}{p-1}$  and  $B$  is the Euler beta function. It is easy to see that the  $L_\infty$ -estimate (1.4) coincides with (1.2) in the particular case of  $p = 2$ .

The aim of this paper is to fill the gap between the cases  $p = 2$ ,  $1 \leq \alpha \leq \infty$ , studied in [19] for  $\mu \geq 0$  on  $[0, T]$  and in [3] for  $\mu$  changing sign, and  $1 < p < \infty$ ,  $\alpha = \infty$ , studied in [6] for  $\mu \geq 0$  on  $[0, T]$ . We will provide sharp  $L_\alpha$ -estimates on the potential  $\mu(t)$  in order to ensure the validity of the anti-maximum principle for problem (1.3) even in the case that  $\mu(t)$  changes sign in  $J$ . Our result extends [19, Corollary 2.5], [3, Theorem 3.2] for arbitrary  $1 < p < \infty$  and [6, Theorem 3.2] for any  $1 \leq \alpha \leq \infty$ , including in this case potentials with zero mean value and solving also the open problem (iii) posed by the authors at the end of [6].

The paper is organized as follows: in Section 2 we introduce some preliminary results needed in Section 3 to prove our main result. In Section 4 we indicate some applications to singular differential equations and offer some remarks and ideas for further research. Finally, Section 5 contains an appendix with the proof of some technical results used in the paper.

## 2. Preliminaries

Throughout the paper, for  $1 \leq \alpha \leq \infty$  and a bounded interval  $[a, b] \subset \mathbb{R}$ , we denote by  $L_\alpha[a, b]$  the usual Lebesgue space with the corresponding norm  $\|\cdot\|_\alpha$ , and by  $\alpha^*$  the conjugate of  $\alpha$  ( $\alpha^* = \frac{\alpha}{\alpha-1}$  if  $\alpha > 1$ ,  $\alpha^* = \infty$  if  $\alpha = 1$  and  $\alpha^* = 1$  if  $\alpha = \infty$ ). For  $x \in L_1[a, b]$  we denote its mean value by  $\bar{x}$ , i.e.

$$\bar{x} = \frac{1}{b-a} \int_a^b x(s) \, ds.$$

Furthermore, for  $x \in L_1[a, b]$  we write  $x \succ 0$  if  $x \geq 0$  a.e. on  $[a, b]$  and  $\bar{x} > 0$ . If  $x \in L_1[a, b]$ , we denote

$$x_* = \inf_{t \in [a, b]} \text{ess } x(t), \quad x^* = \sup_{t \in [a, b]} \text{ess } x(t)$$

and  $x_+(t) = \max\{x(t), 0\}$  for a.e.  $t \in [a, b]$ .

As usual, for an arbitrary subinterval  $I$  of  $\mathbb{R}$  we denote by  $C(I)$  the set of functions  $x: I \rightarrow \mathbb{R}$  which are continuous on  $I$ . For a bounded interval  $J \subset \mathbb{R}$ ,  $C^1(J)$  stands for the set of functions  $x \in C(J)$  with the first derivative continuous on  $J$ . Further,  $AC(J)$  is the set of functions absolutely continuous on  $J$  and  $AC_{loc}(J)$  is the set of functions absolutely continuous on each compact interval  $K \subset J$ . For  $x \in L_\alpha(J)$ ,  $1 \leq \alpha \leq \infty$ , we put

$$\|x\|_{\alpha, J} = \begin{cases} \left( \int_J |x(t)|^\alpha \, dt \right)^{1/\alpha} & \text{if } 1 \leq \alpha < \infty, \\ \sup_{t \in J} \text{ess } |x(t)| & \text{if } \alpha = \infty. \end{cases}$$

If  $1 \leq \alpha \leq \infty$ , then  $W^{1, \alpha}(J)$  denotes the set of functions  $u \in AC(J)$  such that  $u' \in L_\alpha(J)$  and

$$W_0^{1, \alpha} = \{u \in W^{1, \alpha}(J) : u = 0 \text{ on } \partial J\}.$$

Finally, if  $I, J \subset \mathbb{R}$  are subintervals of  $\mathbb{R}$ ,  $J$  bounded, then  $Car(J \times I)$  stands for the set of functions satisfying the Carathéodory conditions on  $J \times I$ , i.e. the set of functions  $f: J \times I \rightarrow \mathbb{R}$  having the following properties: (i) for each  $x \in I$  the function  $f(\cdot, x)$  is measurable on  $J$ ; (ii) for almost every  $t \in J$  the function  $f(t, \cdot)$  is continuous on  $I$ ; (iii) for each compact set  $K \subset I$ , the function  $m_K(t) = \sup_{x \in K} |f(t, x)|$  is

Lebesgue integrable on  $J$ . Solutions of differential equations are in this paper understood in the Carathéodory sense. In particular, a function  $u : J \rightarrow \mathbb{R}$  is a *solution* on the interval  $J$  to the equation

$$(2.1) \quad (\phi_p(u'))' = f(t, u),$$

if  $u \in C^1(J)$ ,  $\phi_p \circ u' \in AC(J)$ ,  $u(t) \in I$  for all  $t \in J$  and

$$(\phi_p(u'(t)))' = f(t, u(t)) \quad \text{for a.e. } t \in J.$$

We will consider various boundary value problems consisting of a differential equation of the form (2.1) and of some additional boundary conditions, like the periodic or the Dirichlet conditions. Their solutions are, as usual, solutions of the differential equation fulfilling the corresponding boundary condition.

DIRICHLET EIGENVALUES. It is known (see [23]) that the eigenvalue problem

$$(2.2) \quad (\phi_p(u'))' + (\lambda + \mu(t)) \phi_p(u) = 0, \quad u = 0 \quad \text{on } \partial J,$$

on a bounded interval  $J \subset \mathbb{R}$  has a sequence of simple eigenvalues

$$-\infty < \lambda_1^D(\mu, J) < \lambda_2^D(\mu, J) < \cdots < \lambda_n^D(\mu, J) < \cdots .$$

In particular, when the potential  $\mu$  is constant ( $\mu(t) \equiv \mu$ ), the eigenvalues are given explicitly by

$$\lambda_n^D(\mu, J) = \left( \frac{n \pi_p}{|J|} \right)^p - \mu, \quad n \in \mathbb{N},$$

where  $|J|$  denotes the length of the interval  $J$  and  $\pi_p$  is defined in (1.5). Using the relationship between Euler's beta and gamma functions, and, in particular, the relations

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} \quad \text{and} \quad \Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}$$

valid for  $x \in (0, 1)$ , we can see that the formula

$$(2.3) \quad \pi_p := \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)}$$

is true, as well.

For  $1 \leq \beta \leq \infty$ ,  $1 < p < \infty$  and an arbitrary closed bounded subinterval  $J$  of  $\mathbb{R}$ , we denote by  $K(\beta, p, J)$  the best Sobolev constant in the inequality

$$C \|u\|_{\beta, J}^p \leq \|u'\|_{p, J}^p \quad \text{for all } u \in W_0^{1,p}(J),$$

that is

$$K(\beta, p, J) = \inf \left\{ \frac{\|u'\|_{p,J}^p}{\|u\|_{\beta,J}^p} : u \in W_0^{1,p}(J) \setminus \{0\} \right\}.$$

Put

$$\varkappa(\beta, p) = \left( \frac{2 \left(1 + \frac{\beta}{p^*}\right)^{1/\beta} B\left(\frac{1}{\beta}, \frac{1}{p^*}\right)}{\beta \left(1 + \frac{p^*}{\beta}\right)^{1/p}} \right)^p \quad \text{for } \beta \in [1, \infty) \text{ and } p \in (1, \infty).$$

It is well known, cf. Talenti [17, p. 357], Zhang [22, Theorem 4.1] or Drábek & Manásevich [9, Theorem 5.1], that

$$(2.4) \quad K(\beta, p, J) = \frac{\varkappa(\beta, p)}{|J|^{p-1+p/\beta}} \quad \text{for } 1 < \beta < \infty.$$

Furthermore, one can show, cf. Lemma 5.1 in Appendix, that the relations

$$(2.5) \quad K(1, p, J) = \frac{\varkappa(1, p)}{|J|^{2p-1}} \quad \text{and} \quad K(\infty, p, J) = \frac{2^p}{|J|^{p-1}}$$

are true, as well. Finally, notice that

$$\begin{aligned} \varkappa(\beta, p) &= \left(\frac{2}{\beta}\right)^p \left(\frac{p^* + \beta}{p^*}\right)^{p/\beta} \left(\frac{\beta}{p^* + \beta}\right) B^p(1/\beta, 1/p^*) \\ &= \left(\frac{2}{\beta}\right)^p \left(\frac{p^* + \beta}{p^*}\right)^{p/\beta} \left(\frac{\beta}{p^* + \beta}\right) \left(\frac{\Gamma(1/\beta) \Gamma(1/p^*)}{\Gamma(1/\beta + 1/p^*)}\right)^p \end{aligned}$$

holds for  $\beta \in [1, \infty)$  and  $p \in (1, \infty)$ .

In [24, Theorem 2.4] the following lower bound for the first Dirichlet eigenvalue  $\lambda_1^D(\mu, J)$  is established in terms of the  $L_\alpha$ -norm of the potential  $\mu(t)$  and of the corresponding best Sobolev constants.

**2.1. Theorem.** *Let  $J$  be a bounded interval in  $\mathbb{R}$ . Furthermore, assume that  $\mu \in L_\alpha(J)$  for some  $1 \leq \alpha \leq \infty$  and  $\|\mu_+\|_{\alpha,J} \leq K(p\alpha^*, p, J)$ . Then*

$$\lambda_1^D(\mu, J) \geq \left(\frac{\pi_p}{|J|}\right)^p \left(1 - \frac{\|\mu_+\|_{\alpha,J}}{K(p\alpha^*, p, J)}\right) \geq 0.$$

**2.2. Remark.** One can check that

$$(2.6) \quad K(p\alpha^*, p, J) = \begin{cases} \frac{2^p}{|J|^{p-1}} & \text{if } \alpha = 1, \\ \frac{\varkappa(p\alpha^*, p)}{|J|^{p-1/\alpha}} & \text{if } 1 < \alpha < \infty, \\ \left(\frac{\pi_p}{|J|}\right)^p & \text{if } \alpha = \infty, \end{cases}$$

where

$$\varkappa(p\alpha^*, p) = \left(\frac{2}{p}\right)^p \left(\frac{\alpha-1}{\alpha}\right)^p \frac{\alpha(p-1)}{(\alpha-1)^{1-1/\alpha}} \left(\frac{1}{p\alpha-1}\right)^{1/\alpha} B^p\left(\frac{\alpha-1}{p\alpha}, \frac{p-1}{p}\right) \text{ for } 1 < \alpha < \infty.$$

Furthermore, for each  $p \in (1, \infty)$ , the function  $\alpha \rightarrow \varkappa(p\alpha^*, p)$  is increasing on  $(1, \infty)$  and  $\lim_{\alpha \rightarrow \infty} \varkappa(p\alpha^*, p) = \pi_p^p$ .

**LOWER AND UPPER FUNCTIONS.** Let  $f \in Car(J \times \mathbb{R})$ . Then a function  $\sigma \in C(J)$  is a *lower function* for problem

$$(2.7) \quad (\phi_p(u'))' = f(t, u), \quad u = 0 \text{ on } \partial J,$$

if  $\sigma \leq 0$  on  $\partial J$  and, for each  $t_0 \in \text{int}(J)$ , either  $\sigma'(t_0-) < \sigma'(t_0+)$ , or there exists an open interval  $J_0 \subset J$  such that  $t_0 \in J_0$ ,  $\sigma \in C^1(J_0)$ ,  $\phi_p \circ \sigma' \in AC(J_0)$  and

$$(\phi_p(\sigma'(t)))' \geq f(t, \sigma(t)) \quad \text{for a.e. } t \in J_0.$$

When all the above inequalities are reversed we call  $\sigma$  an *upper function* for problem (2.7).

Arguing as in the proof of [8, Theorem 5], one can prove the following result, which asserts the solvability of (2.7) in the presence of a pair of well ordered lower and upper solutions.

**2.3. Theorem.** *Assume the existence of a lower function  $\sigma_1$  and an upper function  $\sigma_2$  of problem (2.7) such that  $\sigma_1 \leq \sigma_2$  on  $J$ . Then problem (2.7) has a solution  $u$  such that  $\sigma_1 \leq u \leq \sigma_2$  on  $J$ .*

Analogously to the Dirichlet problem, we define lower and upper functions for the periodic problem

$$(2.8) \quad (\phi_p(u'))' = f(t, u), \quad u(0) = u(T), \quad u'(0) = u'(T).$$

We say that a function  $\sigma \in C(J)$  is a *lower function* for the periodic boundary value problem (2.8) if  $\sigma(0) = \sigma(T)$ ,  $\sigma'(0) \geq \sigma'(T)$  and for each  $t_0 \in \text{int}(J)$ , either

$\sigma'(t_0-) < \sigma'(t_0+)$ , or there exists an open interval  $J_0 \subset \mathbb{R}$  such that  $t_0 \in J_0$ ,  $\sigma \in C^1(J_0)$ ,  $\phi_p \circ \sigma' \in AC(J_0)$  and

$$(\phi_p(\sigma'(t)))' \geq f(t, \sigma(t)) \quad \text{for a.e. } t \in J_0.$$

When all the above inequalities are reversed we call  $\sigma$  an *upper function* for problem (2.8).

### 3. A generalized anti-maximum principle

The following proposition provides sufficient conditions for a Dirichlet problem to be *non-resonant* (that is, the unique solution of the homogeneous Dirichlet problem is the trivial one).

**3.1. Proposition.** *Let  $J = [a, b]$ ,  $-\infty < a < b < \infty$  and  $1 \leq \alpha \leq \infty$ . Furthermore, assume that  $\mu \in L_\alpha[a, b]$  and*

$$\|\mu_+\|_{\alpha, J} \leq K(p\alpha^*, p, J).$$

*Then the Dirichlet problem*

$$(3.1) \quad (\phi_p(u'))' + \mu(t)\phi_p(u) = 0 \quad \text{a.e. on } [t_1, t_2], \quad u(t_1) = u(t_2) = 0,$$

*with  $a \leq t_1 < t_2 \leq b$  has only the trivial solution whenever  $t_2 - t_1 < b - a$ .*

**Proof.** Denote  $\tilde{J} = [t_1, t_2]$  and let  $\tilde{\mu}$  stand for the restriction of  $\mu$  to the interval  $\tilde{J}$ . Since  $0 < t_2 - t_1 < b - a$  we have

$$\|\tilde{\mu}_+\|_{\alpha, \tilde{J}} \leq \|\mu_+\|_{\alpha, J} \leq \frac{\varkappa(p\alpha^*, p)}{(b-a)^{p-1/\alpha}} < \frac{\varkappa(p\alpha^*, p)}{(t_2-t_1)^{p-1/\alpha}} = K(p\alpha^*, p, \tilde{J})$$

and Theorem 2.1 implies that  $\lambda_1^D(\tilde{\mu}, \tilde{J}) > 0$ , which means that (3.1) is nonresonant.  $\square$

**3.2. Remark.** Notice that the conclusion of Proposition 3.1 is true also if

$$\|\mu_+\|_\alpha < K(p\alpha^*, p, J) \quad \text{and} \quad t_2 - t_1 = b - a.$$

**3.3. Definition.** Let  $0 < T < \infty$ . We say that problem (1.3) or

$$(3.2) \quad (\phi_p(u'))' + \mu(t)\phi_p(u) = h(t), \quad u(0) = u(T), \quad u'(0) \geq u'(T),$$

fulfils an *antimaximum principle* if, for each  $h \in L_1[0, T]$  such that  $h \geq 0$  on  $[0, T]$ , any solution of (1.3) or (3.2), respectively, is nonnegative on  $[0, T]$ .

Moreover, we say that problems (1.3) or (3.2) fulfil a *strong antimaximum principle* if they fulfil the anti-maximum principle and, in addition,  $h \succ 0$  implies that each solution  $u$  of problem (1.3) or (3.2), respectively, is positive on  $[0, T]$ .

In the next theorem we present our main result. Notice that, unlike [6, 19], the potential  $\mu(t)$  need not be nonnegative for a.e.  $t \in J$ . Instead we suppose only  $\bar{\mu} \geq 0$  and  $\mu \not\equiv 0$  on  $J$ . In addition, we extend also [3, Theorem 3.2] where only the linear case (i.e.  $p = 2$ ) with the potential having a positive mean value ( $\bar{\mu} > 0$ ) was considered.

**3.4. Theorem.** *Let  $J = [0, T]$ ,  $0 < T < \infty$ ,  $\mu \in L_\alpha(J)$  for some  $\alpha$ ,  $1 \leq \alpha \leq \infty$ ,  $\bar{\mu} \geq 0$  and*

$$(3.3) \quad 0 < \|\mu_+\|_{\alpha, J} \leq K(p\alpha^*, p, J).$$

*Then problem (3.2) fulfils the strong anti-maximum principle.*

**Proof.** Let  $h \in L_1(J)$ ,  $h \geq 0$  a.e. on  $J$  and let  $u$  be an arbitrary solution to (3.2). In particular,  $u \in C^1(J)$ ,  $\phi_p \circ u' \in AC(J)$  and

$$(3.4) \quad (\phi_p(u'(t)))' + \mu(t)\phi_p(u(t)) = h(t) \text{ for a.e. } t \in J,$$

$$(3.5) \quad u(0) = u(T), \quad u'(0) \geq u'(T).$$

Claim 1.  *$u$  does not change its sign on  $J$ .*

Suppose, on the contrary, that

$$\left( \min_{t \in J} u(t) \right) \left( \max_{t \in J} u(t) \right) < 0.$$

Let us extend  $\mu$ ,  $h$  and  $u$  to functions  $T$ -periodic on  $\mathbb{R}$ . Then there are  $a, b, t_1, t_2 \in \mathbb{R}$  such that  $a < b$ ,  $b - a = T$ ,  $a \leq t_1 < t_2 \leq b$ ,  $t_2 - t_1 < b - a$  and

$$u(t_1) = u(t_2) = 0, \quad u > 0 \text{ on } (t_1, t_2).$$

Denote  $\tilde{J} = [a, b]$  and notice that, due to the periodicity of  $\mu$ , we have

$$(3.6) \quad \|\mu_+\|_{\alpha, \tilde{J}} = \|\mu_+\|_{\alpha, J} \quad \text{and} \quad K(p\alpha^*, p, \tilde{J}) = K(p\alpha^*, p, J).$$

In general,  $u$  need not belong to  $C^1[t_1, t_2]$ . However, in any case  $\sigma_1 := u$  is a lower function for the Dirichlet problem (3.1) and it is positive on  $(t_1, t_2)$ .

Further, consider the initial value problem

$$(3.7) \quad (\phi_p(v'))' + \mu(t)\phi_p(v) = 0, \quad v(t_1) = 0, \quad v'(t_1) = 1.$$

By Lemma 5.2 in Appendix this problem has a solution  $v$  defined on the whole  $\mathbb{R}$ . Moreover, due to (3.3) and (3.6), Proposition 3.1 implies that  $v > 0$  on  $(t_1, t_2)$ .

Define  $\sigma_2 := cv$ , with  $c > 0$ , so large that  $\sigma_2 \geq \sigma_1$  on  $[t_1, t_2]$ . Since  $\sigma_2(t_2) > 0$ ,  $\sigma_2$  is an upper function for (3.1). Hence, by Theorem 2.3, problem (3.1) possesses a nontrivial solution. This contradicts Proposition 3.1, which completes the proof of the claim.



*Claim 2.* If  $u \leq 0$  on  $J$ , then  $u \equiv 0$  on  $J$ .

First, assume that  $u < 0$  on  $J$ . Then, dividing equation (3.4) by  $\phi_p(u(t))$  and integrating over  $J$  we arrive (after integration by parts) to the equality

$$(3.8) \quad \begin{cases} (p-1) \int_0^T \left| \frac{u'(t)}{u(t)} \right|^p dt + \int_0^T \mu(t) dt \\ = \int_0^T \frac{h(t)}{\phi_p(u(t))} dt + \left( \frac{\phi_p(u'(0))}{\phi_p(u(0))} - \frac{\phi_p(u'(T))}{\phi_p(u(T))} \right). \end{cases}$$

By our assumptions, both terms on the right-hand side of (3.8) are nonpositive. In particular, we attain a contradiction if  $\bar{\mu} > 0$ .

If  $\bar{\mu} = 0$ , then, having in mind that the right-hand side of (3.8) is certainly non-positive on  $J$ , we deduce that (3.8) can be true only if  $u(t) \equiv u(0) < 0$  and  $h = 0$  a.e. on  $J$ . On the other hand, in this situation, (3.4) reduces to

$$\mu(t) \phi_p(u(0)) = 0 \quad \text{for a.e. } t \in J.$$

By (3.3) and (3.6),  $\mu$  must be nonzero on a subset of  $J$  of a positive measure. This implies  $u(0) = 0$ , a contradiction.

Now, suppose that  $u \leq 0$  on  $J$  and there is a  $t_0 \in J$  such that  $u(t_0) = 0$ . It is easy to see that if  $t_0 \in (0, T)$  then  $u'(t_0) = 0$  must be true. Furthermore, if  $t_0 = 0$ , then, in view of the boundary conditions (3.5), we have  $u(0) = u(T) = 0$ . This implies that the relations  $u'(0) \leq 0 \leq u'(T)$  have to be satisfied, wherefrom, in view of the boundary conditions (3.5), we get easily that the equalities  $u'(0) = 0 = u'(T)$  hold.

Thus, integrating equation (3.4) twice over the interval  $[t_0, T]$ , we derive the equality

$$u(t) = \int_{t_0}^t \phi_p^{-1} \left( - \int_{t_0}^s \mu(\tau) \phi_p(u(\tau)) d\tau + \int_{t_0}^s h(\tau) d\tau \right) ds \quad \text{for } t \in [t_0, T].$$

Therefore,

$$\begin{aligned} |u(t)| = -u(t) &= \int_{t_0}^t \phi_p^{-1} \left( \int_{t_0}^s \mu(\tau) \phi_p(u(\tau)) d\tau - \int_{t_0}^s h(\tau) d\tau \right) ds \\ &\leq \int_{t_0}^t \phi_p^{-1} \left( \int_{t_0}^s |\mu(\tau)| \phi_p(|u(\tau)|) d\tau \right) ds \leq T \phi_p^{-1} \left( \int_{t_0}^t |\mu(\tau)| \phi_p(|u(\tau)|) d\tau \right) \end{aligned}$$

and

$$\phi_p(|u(t)|) \leq \phi_p(T) \int_{t_0}^t |\mu(\tau)| \phi_p(|u(\tau)|) d\tau \quad \text{hold for } t \in [t_0, T].$$

Hence, making use of Gronwall's lemma, we deduce that

$$|u(t)| = 0 \quad \text{for } t \in [t_0, T].$$

In particular,  $u(0) = u(T) = 0$ , and, repeating the above argument on  $[0, t_0]$ , we finally conclude that  $u \equiv 0$  on  $J$ .

Claim 3. Problem (3.2) fulfils the anti-maximum principle.

Indeed, if  $h \geq 0$  a.e. on  $J$  and  $u$  is a solution of (3.2), then, by Claims 1 and 2,  $u \geq 0$  on  $J$ .

Now we are going to prove that the anti-maximum principle is actually strong.

Claim 4. Let  $h \succ 0$  and let  $u$  be a solution of (3.2). Then  $u > 0$  on  $J$ .

By Claim 3 we know that  $u \geq 0$  on  $J$  and, since  $h \succ 0$ ,  $u$  cannot vanish on the whole interval  $J$ . Suppose that there is  $t_0 \in J$  such that  $u(t_0) = 0$ . As in the proof of Claim 1, let us extend  $\mu$ ,  $h$  and  $u$  to functions  $T$ -periodic on  $\mathbb{R}$ . Let  $a, b, t_1, t_2 \in \mathbb{R}$  be such that  $a < b$ ,  $b - a = T$ ,  $a \leq t_1 < t_2 \leq b$ ,  $t_2 - t_1 < b - a$  and

$$u(t_1) = u(t_2) = 0, \quad u > 0 \text{ on } (t_1, t_2).$$

Denote again  $\tilde{J} = [a, b]$ .

If  $0 < t_2 - t_1 < T$ , then the same argument as that used in the proof of Claim 1 leads to a contradiction. Therefore, we have  $t_1 = a$ ,  $t_2 = b$  and, in particular,  $t_2 - t_1 = T$ . Let us recall that the periodic extension of  $u$  need not belong to  $C^1[a, b]$ , as it is possible that  $kT \in (a, b)$  for some  $k \in \mathbb{N} \cup \{0\}$ . If this is the case, then denoting by  $c$  such a point, we obtain  $u'(c-) \leq u'(c+)$  and  $u$  verifies the equality

$$(\phi_p(u'(t)))' + \mu(t) \phi_p(u(t)) = h(t)$$

for a.e.  $t \in [a, c]$  as well as for a.e.  $t \in [c, b]$ . Multiplying this equation by  $u$ , integrating it over  $[a, c]$  and  $[c, b]$  and adding both results, we obtain

$$\begin{aligned} & \int_a^b h(s) u(s) \, ds \\ &= - \int_a^b \phi_p(u'(s)) u'(s) \, ds + \int_a^b \mu(s) \phi_p(u(s)) u(s) \, ds + (\phi_p(u'(c^-)) - \phi_p(u'(c^+))) u(c) \\ &\leq - \int_a^b \phi_p(u'(s)) u'(s) \, ds + \int_a^b \mu(s) \phi_p(u(s)) u(s) \, ds. \end{aligned}$$

Furthermore, having in mind the definition of  $K(\beta, p, J)$ , and applying Hölder's inequality, we get

$$\begin{aligned} 0 &\leq \int_a^b h(s) u(s) \, ds \leq -\|u'\|_{p, \tilde{J}}^p + \|\mu_+\|_{\alpha, \tilde{J}} \|u\|_{\alpha^* p, \tilde{J}}^p \\ &\leq \left( \|\mu_+\|_{\alpha, \tilde{J}} - K(p \alpha^*, p, \tilde{J}) \right) \|u\|_{\alpha^* p, \tilde{J}}^p = (\|\mu_+\|_{\alpha, J} - K(p \alpha^*, p, J)) \|u\|_{\alpha^* p, J}^p \leq 0. \end{aligned}$$

Since  $u > 0$  on  $(a, b)$ , this is possible only if  $h \equiv 0$  on  $[a, b]$ , i.e.  $h \equiv 0$  on  $J$ , which contradicts our assumption  $h > 0$ .  $\square$

Notice that, arguing as in the Claim 4 of the previous proof, we can derive also the following result.

**3.5. Corollary.** *Under the conditions of Theorem 3.4,  $u > 0$  on  $[0, T]$  holds for each solution  $u$  on  $[0, T]$  of the equation*

$$(\phi_p(u'))' + \mu(t) \phi_p(u) = 0$$

fulfilling the boundary conditions  $u(0) = u(T)$ ,  $u'(0) > u'(T)$ .

**3.6. Example.** Let us consider the sign-changing potential  $\mu(t) = a(1 + b \cos(t))$ , where  $a > 0$  and  $b \in \mathbb{R}$ , and the problem

$$(3.9) \quad (\phi_p(x'))' + a(1 + b \cos(t)) \phi_p(x) = h(t), \quad x(0) = x(2\pi), \quad x'(0) \geq x'(2\pi).$$

For  $p = 2$ ,  $b = 1$  (i.e. the potential  $\mu$  is nonnegative) and periodic boundary conditions (i.e.  $x'(0) = x'(2\pi)$ ), it is known that (3.9) satisfies the strong anti-maximum principle when  $0 < a < 0.16448$ , which is a key ingredient in [18, Corollary 4.8] to ensure the solvability of the Brillouin electron beam-focusing equation

$$x'' + a(1 + \cos(t))x = \frac{1}{x^\lambda}, \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi).$$

(For more information about this problem, see e.g. [1] or [21, Example 4.4].)

Our Theorem 3.4 ensures that problem (3.9) satisfies the strong anti-maximum principle provided that  $a > 0$  and

$$(3.10) \quad 0 < a \|(1 + b \cos(t))_+\|_{\alpha, [0, 2\pi]} \leq K(p\alpha^*, p, [0, 2\pi]) \quad \text{for some } \alpha \in [1, \infty).$$

So, for fixed  $p > 1$  and  $b \in \mathbb{R}$  condition (3.10) is satisfied when

$$0 < a < M(p, b) := \sup_{\alpha \in [1, \infty]} \frac{K(p\alpha^*, p, [0, 2\pi])}{\|(1 + b \cos(t))_+\|_{\alpha, [0, 2\pi]}}.$$

Figure 3.6 was obtained by means of the software system Mathematica. It gives the graph of the function  $M(p, \cdot)$  for several fixed values of  $p$ . Let us recall that  $M(2, 0) = 1/4$  and  $M(2, 1) \doteq 0.16448$ .

**3.7. Remark.** Theorem 3.4 generalizes [3, Theorem 3.2] where the authors proved that problem (1.3) fulfils the strong anti-maximum principle whenever  $p = 2$ ,  $\bar{\mu} > 0$  and  $\|\mu_+\|_{\alpha, J} < K(2\alpha^*, 2, J)$  for some  $\alpha$ ,  $1 \leq \alpha \leq \infty$ . Actually, in [3, Theorem 3.2] it

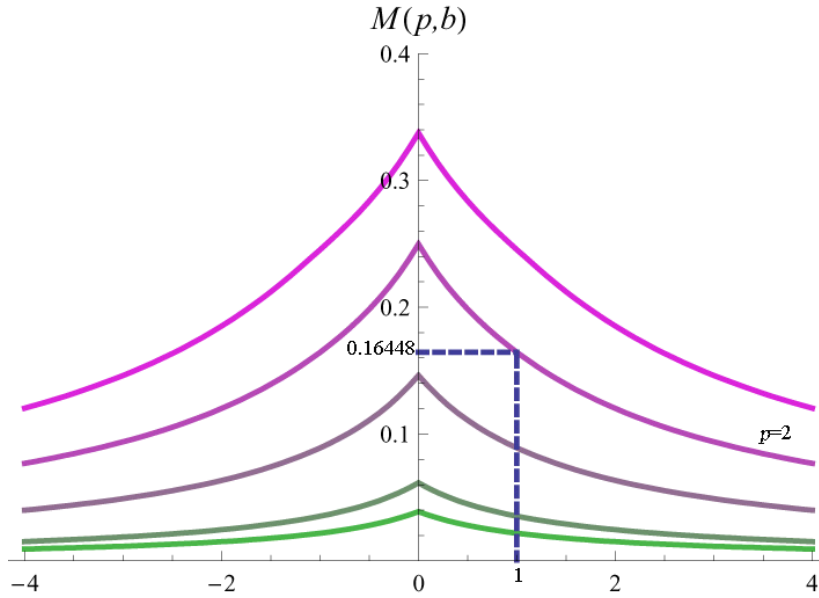


Figure 1: Graph of  $M(p, b)$  for  $p = 1.5, 2, 2.7, 3.7, 4.2$  (from above to below)

is proved that under the above conditions problem (1.3) has a unique solution given by the expression

$$(3.11) \quad u(t) = \int_0^T G(t, s) h(s) ds, \quad t \in J,$$

where  $G$  is the corresponding Green's function and  $G$  is positive on  $J \times J$ . Moreover, one can see in [18] that problem (1.3) with  $p = 2$  fulfils the anti-maximum principle if and only if  $G$  is nonnegative on  $J \times J$ .

On the other hand, the particular choice of the constant potential  $\mu = (\pi/|J|)^2$  (for which  $\|\mu_+\|_{\infty, J} = (\pi/|J|)^2 = K(2, 2, J)$ ) shows that it is not possible to guarantee the strict positivity of  $G$  on  $J \times J$  when equality is attained in the norm estimate (see [2, Lemma 2.5]). In general, for  $p = 2$  it follows from (3.11) that if problem (1.3) fulfils the strong anti-maximum principle then, for any  $t \in J$ , the function  $G(t, \cdot)$  vanishes on a (possibly empty) set of Lebesgue measure zero. Since  $u'' + \mu(t)u$  coupled with the periodic boundary value conditions generates a self-adjoint operator, Green's function  $G$  is symmetric and, as a consequence, the zeroes of  $G(\cdot, s)$  form a set of Lebesgue measure zero, as well.

## 4. Final comments

### 4.1. Applications to singular periodic problems

It is known that the antimaximum principle is a useful tool in the study of nonlinear periodic problems having weak repulsive singularity, see e.g. [6], [13, Section 5.4], [14, Section 8.4] or [16]. In particular, in [6] some existence results for problem (2.8) are established. They rely on the comparison of (2.8) with the related quasilinear problem having a nonnegative essentially bounded potential and fulfilling the antimaximum principle. Contrary to [6], [13], [14] or [16], the antimaximum principle we have delivered in this paper (see Theorem 3.4) concerns also the case of a potential from  $L_\alpha[0, T]$  which may change its sign. Therefore, one can see that the results from [6] or [14] can be extended to such a case. For instance, the principles [6, Theorems 4.2 and 4.3] may be now reformulated as follows.

Let  $r > 0$ ,  $A \geq r$ ,  $B \geq A$ . Furthermore, let  $\beta \in L_1[0, T]$  be such that  $\bar{\beta} \leq 0$  (with  $\bar{\beta} < 0$  if  $1 < p < 2$ ), let  $\mu$  verify the assumptions of Theorem 3.4 for some  $\alpha$ ,  $1 \leq \alpha \leq \infty$ , and let the relations

$$(4.1) \quad f(t, x) \leq \beta(t) \quad \text{for a.e. } t \in [0, T] \quad \text{and all } x \in [A, B],$$

$$(4.2) \quad f(t, x) + \mu(t) \phi_p(x - r) \geq 0 \quad \text{for a.e. } t \in [0, T] \quad \text{and all } x \in [r, B]$$

and

$$(4.3) \quad B - A \geq \frac{T}{2} \|m\|_1^{p^* - 1},$$

where

$$(4.4) \quad m(t) = \max \{ \sup \{ f(t, x) : x \in [r, A] \}, \beta(t), 0 \} \quad \text{for a.e. } t \in [0, T]$$

be satisfied. Then problem (2.8) has a solution  $u$  such that

$$(4.5) \quad r \leq u \leq B \quad \text{on } [0, T] \quad \text{and} \quad \|u'\|_\infty < \phi_p^{-1}(\|m\|_1).$$

### 4.2. Further research

1.- *Monotone method.* For the linear case ( $p = 2$ ), besides the existence result expressed in the previous theorem, it is possible to perform a monotone iteration as in [19, Theorem 3.2], starting at the lower (upper) solution and converging to the minimal (maximal) solution. However, for the nonlinear case,  $p \neq 2$ , Theorem 3.4 is not strong enough to develop the monotone method. In order to succeed, as one can see in [4] for

a Neumann boundary value problem, we would need the following strong version:

$$\left. \begin{aligned} (\phi_p(u'))' + \mu(t) \phi_p(u) &\geq (\phi_p(v'))' + \mu(t) \phi_p(v), \\ u(a) - u(b) = 0 &= v(a) - v(b), \\ u'(a) - u'(b) &\geq 0 \geq v'(a) - v'(b). \end{aligned} \right\} \implies u \geq v.$$

2.- *Nonhomogeneous problems.* It would be interesting to give an anti-maximum principle for the equation

$$(4.6) \quad (\phi_p(u'(t)))' + \mu(t) \phi_q(u(t)) = h(t), \quad t \in [a, b], \quad 1 \leq p, q \leq \infty,$$

together with different kinds of boundary conditions. The set of eigenvalues and eigenfunctions for the equation

$$\phi_p(u'(t))' + \mu \phi_q(u(t)) = 0,$$

with a constant  $\mu \in \mathbb{R}$  and Dirichlet, Neumann or periodic boundary conditions, has been described in [9]. However, as far as the authors are aware, anti-maximum principles have been studied only for equation (4.6) with Neumann boundary conditions,  $1 \leq p \leq \infty$  and  $q = 2$  in [4].

3.- *Sign-changing potential.* In Theorem 3.4 we deal with an indefinite potential  $\mu(t)$  (i.e.,  $\mu(t)$  can change sign) such that  $\bar{\mu} > 0$ . Anti-maximum principles for the  $p$ -Laplacian under Neumann or Dirichlet boundary conditions with indefinite potential have been investigated in the past decades (see [11] and references therein).

## 5. Appendix

**5.1. Lemma.** *The relations (see (2.5))*

$$K(1, p, J) = \frac{\varkappa(1, p)}{|J|^{2p-1}} \quad \text{and} \quad K(\infty, p, J) = \frac{2^p}{|J|^{p-1}}$$

hold for each  $p$ ,  $1 < p < \infty$ , and each bounded interval  $J \subset \mathbb{R}$ .

**Proof.** Recall that, by [17],  $K(\beta, p, J)$  is given by (2.4) if  $1 < \beta < \infty$ .

Case  $\beta = 1$ . Letting  $\beta \rightarrow 1$  in the relation

$$(5.1) \quad K(\beta, p, J) \|u\|_{\beta, J}^p \leq \|u'\|_{p, J}^p \quad \text{for all } u \in W_0^{1, p}(J),$$

we obtain

$$(5.2) \quad \frac{\varkappa(1, p)}{|J|^{2p-1}} \|u\|_{1, J}^p \leq \|u'\|_{p, J}^p \quad \text{for all } u \in W_0^{1, p}(J).$$

Moreover, as noticed in [24, Remark 2.2 (i)], the equality in (5.2) is achieved by some  $u \in W_0^{1,p}(J)$ . This proves the equality  $K(1, p, J) = \frac{\varkappa(1, p)}{|J|^{2p-1}}$ .

Case  $\beta = \infty$ . First, recall that  $\lim_{\beta \rightarrow \infty} \|u\|_{\beta, J}^p = \|u\|_{\infty, J}^p$  holds for all  $u \in W_0^{1,p}(J)$  and all  $p \in (1, \infty)$  (cf. e.g. [20, Theorem I.3.1]). Therefore, letting  $\beta \rightarrow \infty$  in (5.1), we get

$$\frac{2^p}{|J|^{p-1}} \|u\|_{\infty, J}^p \leq \|u'\|_{p, J}^p \quad \text{for all } u \in W_0^{1,p}(J).$$

On the other hand, let  $C > 0$  be an arbitrary constant such that

$$C \|u\|_{\infty, J}^p \leq \|u'\|_{p, J}^p \quad \text{for all } u \in W_0^{1,p}(J).$$

Since  $\|u\|_{\beta, J}^p \leq \|u\|_{\infty, J}^p |J|^{p/\beta}$ , we have

$$\frac{C}{|J|^{p/\beta}} \|u\|_{\beta, J}^p \leq C \|u\|_{\infty, J}^p \leq \|u'\|_{p, J}^p \quad \text{for all } u \in W_0^{1,p}(J),$$

and, by the definition of  $K(\beta, p, J)$ , it follows that

$$\frac{C}{|J|^{p/\beta}} \leq K(\beta, p, J).$$

Thus, letting  $\beta \rightarrow \infty$ , we get

$$C \leq \frac{2^p}{|J|^{p-1}},$$

wherefrom the equality  $K(\infty, p, J) = \frac{2^p}{|J|^{p-1}}$  immediately follows.  $\square$

The following lemma is essentially [10, Proposition 3.2], where the assumption on the positivity of the potential  $\mu$  was needed only to show the uniqueness of the solution obtained. We include the proof for the convenience of the reader.

**5.2. Lemma.** *Suppose that  $\mu \in L_{1,loc}(\mathbb{R})$ . Then for each  $t_0 \in \mathbb{R}$  and  $\tilde{x}, \tilde{y} \in \mathbb{R}$ , the initial value problem*

$$(5.3) \quad (\phi_p(u'))' + \mu(t) \phi_p(u) = 0, \quad u(t_0) = \tilde{x}, \quad u'(t_0) = \tilde{y}$$

*possesses at least one solution defined on the whole  $\mathbb{R}$ .*

**Proof.** The existence of a local solution of (5.3) defined on the interval  $(t_0 - \delta, t_0 + \delta)$  for some  $\delta > 0$  is a direct consequence of the classical Carathéodory theory for ordinary differential equations.

Assume that  $u$  is a solution to (5.3) on  $(a_1, b_1)$  and  $t_0 \in (a_1, b_1)$ . To prove that this solution can be extended to the whole  $\mathbb{R}$ , it suffices to show that there are constants  $M_0, M_1 \in (0, \infty)$  such that the relations

$$(5.4) \quad |u(t)| \leq M_0 \quad \text{and} \quad |u'(t)| \leq M_1$$

are true for all  $t \in (a_1, b_1)$ . To this aim, assume that an arbitrary  $t \in [t_0, b_1)$  is given. Integrating the differential equation in (5.3) over  $[t_0, t]$ , we obtain

$$(5.5) \quad |u'(t)|^{p-1} \leq |\tilde{y}|^{p-1} + v(t)$$

and

$$(5.6) \quad |u(t)| \leq |\tilde{x}| + \int_{t_0}^t (|\tilde{y}|^{p-1} + v(s))^{\frac{1}{p-1}} ds$$

where

$$(5.7) \quad v(t) = \int_{t_0}^t |\mu(s)| |u(s)|^{p-1} ds.$$

Furthermore, as  $(x + y)^{p-1} \leq d(p)(x^{p-1} + y^{p-1})$  holds for  $x, y \in [0, \infty)$ , where  $d(p) = 1$  if  $1 < p < 2$  and  $d(p) = 2^{p-2}$  if  $2 \leq p < \infty$  (cf. e.g. [12, Section VIII.4.2]), we get

$$\begin{aligned} |u(t)|^{p-1} &\leq \left( |\tilde{x}| + \int_{t_0}^t (|\tilde{y}|^{p-1} + v(s))^{\frac{1}{p-1}} ds \right)^{p-1} \\ &\leq \left( |\tilde{x}| + d(p^*) \int_{t_0}^t (|\tilde{y}| + v(s)^{\frac{1}{p-1}}) ds \right)^{p-1} \\ &\leq d(p) \left( (|\tilde{x}| + (b_1 - a_1) d(p^*) |\tilde{y}|)^{p-1} + ((b_1 - a_1) d(p^*))^{p-1} v(t) \right) \leq C(1 + v(t)) \end{aligned}$$

where

$$C = \max\{d(p) (|\tilde{x}| + (b_1 - a_1) d(p^*) |\tilde{y}|)^{p-1}, d(p) ((b_1 - a_1) d(p^*))^{p-1}\}.$$

To summarize, we have

$$(5.8) \quad |u(t)|^{p-1} \leq C(1 + v(t)) \quad \text{for all } t \in [t_0, b_1).$$

Therefore, differentiating (5.7), we find that the inequality

$$v'(t) \leq C |\mu(t)| (1 + v(t))$$

holds for a.e.  $t \in [t_0, b_1)$ . Hence, by Gronwall's inequality, there is a constant  $\tilde{C} \in (0, \infty)$  such that  $v(t) \leq \tilde{C}$  for all  $t \in [t_0, b_1)$ . Now, by (5.8) and (5.5), we conclude that

$$|u(t)| \leq (C(1 + \tilde{C}))^{\frac{1}{p-1}} \quad \text{and} \quad |u'(t)| \leq (|\tilde{y}|^{p-1} + \tilde{C})^{\frac{1}{p-1}} \quad \text{for all } t \in [t_0, b_1),$$

i.e. (5.4) is true for  $t \in [t_0, b_1)$ . Analogously, we could show that (5.4) is true also for  $t \in (a_1, t_0]$ .  $\square$

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