

On extremal fixed points in Schauder's
theorem with applications to differential
equations

J. Ángel Cid

Departamento de Análise Matemática,
Facultade de Matemáticas, Campus Sur,
Universidade de Santiago de Compostela,
15782 Santiago de Compostela, Spain

E-mail : joseangel.cid@rai.usc.es

Abstract

We derive necessary and sufficient conditions for the existence of a least and a greatest fixed point of an operator which satisfies the hypothesis of Schauder's theorem. The so obtained results are applied to prove existence of extremal solutions for some initial and boundary value problems.

1 Introduction

In an ordered normed space it is well known that a nondecreasing completely continuous selfmap of a given order interval has a least and a greatest fixed point. In general, this result is not true if we do not assume any monotonicity condition. The aim of this paper is to characterize the existence of the extremal fixed points for a completely continuous operator T which satisfies Schauder's theorem. The statement of this result, which is given in section 2, is as follows: T has a greatest fixed point if and only if the set of fixed points of T is upward directed (i.e. for any pair of fixed points there exists another one which is greater than both of them). We point out that in recent years the directness has played a crucial role to prove the existence of extremal solutions in the framework of nonlinear elliptic and parabolic problems (see the monograph [1]).

In section 3 we present two illustrative applications of our results. The first one is a new and shorter proof of the existence of extremal solutions for the scalar first order initial value problem with Carathéodory functions. In the second one we give a proof which is simpler than that obtained in the monograph [3] for the existence of extremal solutions, between assumed lower and upper solutions, for a second order periodic boundary value problem.

2 Abstract results

We say that a subset Y of a partially ordered set (poset) X is *upward directed* if for each pair $y_1, y_2 \in Y$ there exists $y_3 \in Y$ such that $y_1 \leq y_3$ and $y_2 \leq y_3$. Analogously, Y is *downward directed* if for each pair $y_1, y_2 \in Y$ there exists $y_3 \in Y$ such that $y_3 \leq y_1$ and $y_3 \leq y_2$.

A poset X is a *lattice* if $x_1 \vee x_2 := \sup\{x_1, x_2\}$ and $x_1 \wedge x_2 := \inf\{x_1, x_2\}$, exist for all $x_1, x_2 \in X$. Every totally ordered set is a lattice and every lattice is upward and downward directed. A lattice X is *complete* when each non empty subset $B \subset X$ has supremum, denoted by $\bigvee B$, and infimum, denoted by $\bigwedge B$. In particular, every complete lattice has the maximum and the minimum.

Let N be a normed space. A subset $K \subset N$ is a *cone* if it is closed, $K + K \subset K$, $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. A cone K yields a partial ordering in N given by $x \leq y$ if and only if $y - x \in K$. N is an *ordered normed space* if N is ordered by a cone.

In an ordered normed space N we have that the intervals

$$[x] := \{z \in X : z \leq x\} \quad \text{and} \quad]x := \{z \in X : x \leq z\}$$

are closed for all $x \in N$, because $[x] = x - K$ and $]x = x + K$.

An operator $T : D \subset N \rightarrow N$ is called *completely continuous* if it is continuous and moreover $\overline{T(M)}$ is a compact set whenever $M \subset D$ is bounded. We say that $x^* \in D$ is the *greatest fixed point* of T if x^* is a fixed point of T and if $x \leq x^*$ for any other fixed point $x \in D$. The *least fixed point* is defined similarly by reversing the inequality. When both, the least and the greatest fixed point of T , exist we call them

extremal fixed points.

The following theorem is our main result.

Theorem 2.1 *Let N be an ordered normed space, $D \subset N$ a non empty, bounded, closed and convex subset and $T : D \rightarrow D$ a completely continuous operator. Then the set of fixed points of T*

$$P = \{x \in D : Tx = x\},$$

is compact and non empty. Moreover the following claims hold:

- i) T has a greatest (least) fixed point if and only if P is upward (downward) directed.*
- ii) If P is a lattice then P is a complete lattice.*

Proof. Schauder's fixed point theorem ensures that P is non empty. Moreover, P is closed, because $P = (T - Id)^{-1}(0)$, and since $\overline{T(D)}$ is a compact set and $P = T(P) \subset \overline{T(D)}$ we deduce that P is also compact.

Proof of i). If T has a greatest fixed point then obviously P is upward directed. Conversely, suppose that P is upward directed. Then the following family of closed subsets of P

$$\mathcal{F}_1 = \{[x] \cap P : x \in P\}$$

has the finite intersection property. Since P is compact we have that $\bigcap_{x \in P} ([x] \cap P)$ contains a point x^* , which is a greatest fixed point of T because $x^* \in P$ and $x^* \in [x]$, i.e. $x^* \geq x$, for all $x \in P$. By using dual arguments we prove that T has a least fixed point if and only if P is downward directed.

Proof of ii). Suppose that P is a lattice and let $B \subset P$ be a non empty subset. Since P is upward directed we know, by claim *i*), that T has a greatest fixed point x^* . Therefore the following family of closed subsets of P

$$\mathcal{F}_2 = \{[x, u] \cap P : x \in B, u \in P \text{ is an upper bound of } B\},$$

is non empty, because $x^* \in P$ is an upper bound of B . Moreover \mathcal{F}_2 has the finite intersection property because

$$\bigcap_{i=1}^n \{x_i : i = 1, \dots, n\} \in \bigcap_{i=1}^n ([x_i, u_i] \cap P),$$

for any $[x_i, u_i] \cap P \in \mathcal{F}_2$, $i \in \{1, \dots, n\}$. Then, since P is compact, the intersection of all sets of the family \mathcal{F}_2 contains a point, which by construction is the supremum of B in P . By using dual arguments we prove that there exists the infimum of B in P and thus P is a complete lattice. \square

A list of different general conditions which imply that an upward directed set has the maximum can be founded in section 5 in [4].

3 Applications: existence of extremal solutions of differential equations

3.1 A first order initial value problem

Let $I = [0, T]$, with $T > 0$. We say that $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a *Carathéodory function* if for all $x \in \mathbb{R}$ the function $f(\cdot, x)$ is measurable, for a.a. $t \in I$ the function $f(t, \cdot)$ is continuous and moreover

there exists $m \in L^1(I)$ such that

$$|f(t, x)| \leq m(t) \quad \text{for a.a. } t \in I \text{ and for all } x \in \mathbb{R}.$$

In case $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function by a solution of problem

$$x'(t) = f(t, x(t)) \quad \text{for a.a. } t \in I, \quad x(0) = x_0, \quad (3.1)$$

we mean an absolutely continuous function $x : I \rightarrow \mathbb{R}$ such that $x(0) = x_0$ and that satisfies the differential equation for almost all $t \in I$. If x_{max} is a solution of (3.1) and for any other solution x we have that

$$x(t) \leq x_{max}(t) \quad \text{for all } t \in I,$$

we say that x_{max} is the *maximal solution* of (3.1). The concept of *minimal solution* x_{min} is defined in a similar way by reversing the inequality. When both the maximal and the minimal solution, exist we call them *extremal solutions* of (3.1).

It is well known that problem (3.1) has extremal solutions (see [2]). Next we give a new and shorter proof of this fact using theorem 2.1.

Theorem 3.1 *Suppose $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function. Then the solution set*

$$\mathcal{S} = \{x : I \rightarrow \mathbb{R} : x \text{ is a solution of (3.1)}\},$$

is a non empty compact subset of $\mathcal{C}(I)$. Moreover \mathcal{S} is a complete lattice and in particular problem (3.1) has extremal solutions.

Proof. Clearly \mathcal{S} matches up the set of fixed points P of operator $T : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$ defined for each $x \in \mathcal{C}(I)$ as

$$Tx(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds \quad \text{for all } t \in I.$$

It is easy to prove that T is completely continuous and bounded. Therefore it follows from theorem 2.1 that $P = \mathcal{S}$ is a non empty compact subset of $\mathcal{C}(I)$. Moreover, $\mathcal{C}(I)$ with the cone of all nonnegative functions is an ordered normed space and $\mathcal{S} = P$ is a lattice because the maximum and the minimum (pointwise) of solutions of (3.1) is also a solution of (3.1). Therefore, from theorem 2.1 *ii*) it follows that $P = \mathcal{S}$ is a complete lattice and in particular the extremal solutions of (3.1) exist. \square

3.2 A periodic boundary value problem

We consider the second order periodic problem

$$u''(t) = f(t, u(t)), \quad u(a) = u(b), \quad u'(a) = u'(b), \quad (3.2)$$

where $a < b$ and f is continuous.

We define the concept of lower and upper \mathcal{C}^2 -solutions of problem (3.2) following [3]: a function $\alpha \in \mathcal{C}([a, b])$ such that $\alpha(a) = \alpha(b)$ is a \mathcal{C}^2 -lower solution of problem (3.2) if its periodic extension on \mathbb{R} , defined by $\alpha(t) = \alpha(t + b - a)$, is such that for any $t_0 \in \mathbb{R}$ either $D_- \alpha(t_0) < D^+ \alpha(t_0)$, or there exist an open interval I_0 with $t_0 \in I_0$ and a function $\alpha_0 \in \mathcal{C}^1(I_0, \mathbb{R})$ such that:

- (i) $\alpha(t_0) = \alpha_0(t_0)$ and $\alpha(t) \geq \alpha_0(t)$ for all $t \in I_0$;
- (ii) $\alpha_0''(t_0)$ exists and $\alpha_0''(t_0) \geq f(t_0, \alpha_0(t_0))$.

A function $\beta \in \mathcal{C}([a, b])$ such that $\beta(a) = \beta(b)$ is a \mathcal{C}^2 -upper solution of problem (3.2) if its periodic extension on \mathbb{R} is such that for any $t_0 \in \mathbb{R}$ either $D^-\beta(t_0) > D_+\beta(t_0)$, or there exist an open interval I_0 with $t_0 \in I_0$ and a function $\beta_0 \in \mathcal{C}^1(I_0, \mathbb{R})$ such that:

- (i) $\beta(t_0) = \beta_0(t_0)$ and $\beta(t) \leq \beta_0(t)$ for all $t \in I_0$;
- (ii) $\beta_0''(t_0)$ exists and $\beta_0''(t_0) \leq f(t_0, \beta_0(t_0))$.

The following result concerning lower and upper \mathcal{C}^2 -solutions holds (propositions 2.1 and 2.2 in [3]).

Proposition 3.2 *Let α_1 and α_2 be \mathcal{C}^2 -lower solutions. Then*

$$\alpha(t) = \alpha_1 \vee \alpha_2(t) := \max\{\alpha_1(t), \alpha_2(t)\} \quad \text{for all } t \in [a, b],$$

is a \mathcal{C}^2 -lower solution.

Let β_1 and β_2 be \mathcal{C}^2 -upper solutions. Then

$$\beta(t) = \beta_1 \wedge \beta_2(t) = \min\{\beta_1(t), \beta_2(t)\} \quad \text{for all } t \in [a, b],$$

is a \mathcal{C}^2 -upper solution.

Habets and De Coster prove the existence of extremal solutions of (3.2) between α and β (theorem 2.4 of [3]) by using Akô's method. We are going to give a simpler and shorter proof based on theorem 2.1.

Theorem 3.3 *Let α and β be C^2 -lower and upper solutions of (3.2), such that $\alpha \leq \beta$, define $E = \{(t, u) \in [a, b] \times \mathbb{R} : \alpha(t) \leq u \leq \beta(t)\}$ and assume that f is continuous on E .*

Then the solution set

$$\mathcal{S} = \{u \in C^2([a, b]) : \alpha \leq u \leq \beta, u \text{ is a solution of (3.2)}\},$$

is a non empty compact subset of $C([a, b])$. Moreover, there exist the maximal, u_{max} , and the minimal, u_{min} , solutions of problem (3.2) between α and β , that is, if $u \in \mathcal{S}$ then

$$u_{min}(t) \leq u(t) \leq u_{max}(t) \quad \text{for all } t \in [a, b].$$

Proof. In theorem 2.3 in [3] the authors prove that \mathcal{S} equals the set of fixed points P of operator $T : C([a, b]) \rightarrow C([a, b])$ defined for each $u \in C([a, b])$ as

$$Tu(t) = \int_a^b G(t, s)(f(t, \gamma(s, u(s))) - \gamma(s, u(s)))ds,$$

where $G(t, s)$ is the Green function that corresponds to problem

$$u''(t) - u(t) = f(t), \quad u(a) = u(b), \quad u'(a) = u'(b),$$

and where $\gamma : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\gamma(t, u) = \begin{cases} \beta(t), & \text{if } u > \beta(t), \\ u, & \text{if } \alpha(t) \leq u \leq \beta(t), \\ \alpha(t), & \text{if } u < \alpha(t). \end{cases}$$

Since T is completely continuous and bounded, we deduce by theorem 2.1 that $P = \mathcal{S}$ is a non empty compact subset of $C([a, b])$.

Now, we are going to prove that $\mathcal{S} = P$ is upward directed with respect to the order induced in $\mathcal{C}([a, b])$ by the cone of nonnegative functions. For given $u_1, u_2 \in \mathcal{S}$ proposition 3.2 ensures that $\alpha_1 := u_1 \vee u_2 \leq \beta$ is a \mathcal{C}^2 -lower solution of (3.2) and then, repeating the above argument, there exists a solution u_3 of (3.2) between α_1 and β . Then, since $u_3 \in \mathcal{S}$, $u_1 \leq u_3$ and $u_2 \leq u_3$, it follows that $\mathcal{S} = P$ is upward directed.

Now, from theorem 2.1 *i)*, we deduce the existence of a greatest fixed point u_{max} of T , which is the maximal solution of (3.2) in the sector enclosed by α and β . By using a dual argument we prove that problem (3.2) has the minimal solution between α and β . \square

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References

- [1] Carl, S. and Heikkilä, S., “ Nonlinear Differential Equations in Ordered Spaces”, Chapman & Hall / CRC, Boca Ratón, 2000.
- [2] Coddington, E. A. and Levinson, N., *Theory of ordinary differential equations*, McGraw-Hill, New York-Toronto-London, (1955).
- [3] Habets, P. and De Coster, C., *The method of lower and upper solutions in the study of boundary value problems*, to be published.

- [4] Heikkilä, S., Applications of a recursion method to maximization problems in partially ordered spaces, *Nonlinear Stud.*, 3 (1996), no. 2, 249–260.