SOLVABILITY OF THE FORCED RELATIVISTIC PENDULUM WITH A DERIVATIVE DEPENDENT COEFFICIENT

PABLO AMSTER[1] AND J. ÁNGEL CID[2,3]

Abstract: We deal with the solvability of the forced pendulum in the relativistic regime. As a novelty with respect to most references in the mathematical literature, and based on an alternative framework discussed in some physical papers, we take into account also the relativistic effects in the forces acting on the pendulum which lead to the study of a $\varphi-$laplacian equation with an unbounded derivative dependent term. In this new relativistic setting we recover several results about the structure of the solvability set that are well–known for the classical pendulum. We also show that the solvability set for the classical pendulum is the limit of the relativistic one when the speed of light blows up to infinity.

Keywords: Periodic solution, pendulum equation, special relativity, Leray-Schauder degree.

Mathematics Subject Classification: 34C25, 83A05, 47H11.

1. Introduction

The forced pendulum equation is a deceptively simple model that has inspired research for more than a century and has been considered by Jean Mawhin as a "paradigm for Nonlinear Analysis", see [15, 16]. The pendulum is a mechanical device that consists in a bob with mass $m > 0$ subject to a rod of length $l > 0$ and swinging under the effect of a uniform gravitational field g, a tangential damping proportional to the velocity (with constant $\mathfrak{K} \geq 0$) and, eventually, to an external forcing torque $\mathfrak{f}(t)$. In this framework, Newton's second law provides the well-known equation for the angular displacement $u(t)$ of the bob: the derivative of the linear momentum equals the forces acting on it, that is,

$$\frac{d}{dt}\left(m\,l\,u'(t)\right) = -g\,m\,\sin(u(t)) - \mathfrak{K}\,l\,u'(t) + \mathfrak{f}(t). \tag{1}$$

When the forcing $\mathfrak{f}(t)$ is periodic, with period $T > 0$, a natural question is the existence of some T-periodic solution $u(t)$ of (1). Even such a seemingly easy question is not completely understood although the structure of the "solvability"

The authors were partially supported by the Agencia Estatal de Investigación (Ministerio de Ciencia e Innovación), Spain, and European Regional Development Fund, Project PID2021-122442NB–100.

set is known, [1, 16]: writing $\mathfrak{f}(t) = \tilde{\mathfrak{f}}(t) + \alpha$, where $\tilde{\mathfrak{f}}$ has mean value zero, and defining $\mathfrak{I}_{\tilde{\mathfrak{f}}}$ as the set of values α for which equation (1) with $\mathfrak{f}(t) = \tilde{\mathfrak{f}}(t) + \alpha$ has a T-periodic solution, then

(P1) $\mathfrak{I}_{\tilde{\mathfrak{f}}} = [d_{\tilde{\mathfrak{f}}}, D_{\tilde{\mathfrak{f}}}]$ for some $d_{\tilde{\mathfrak{f}}} \leq D_{\tilde{\mathfrak{f}}}$.
(P2) $d_{\tilde{\mathfrak{f}}}$ and $D_{\tilde{\mathfrak{f}}}$ depend continuously on $\tilde{\mathfrak{f}}$.
(P3) Equation (1) has at least two T-periodic solutions (geometrically different, i.e., not differing in a multiple of 2π) if $\alpha \in (d_{\tilde{\mathfrak{f}}}, D_{\tilde{\mathfrak{f}}})$.

Whether could be $d_{\tilde{\mathfrak{f}}} = D_{\tilde{\mathfrak{f}}}$ or not for some $\tilde{\mathfrak{f}}$ is called the *degeneracy problem* and it is still an important open problem in the field, [19].

A recent and fruitful line of research on the pendulum equation subject to relativistic effects was initiated in [20], provoking and important impact in the literature [2, 3, 4, 5, 6, 7, 8, 21, 22], by replacing the left–hand side of (1) by the derivative of the relativistic linear momentum, [11], namely

$$\frac{d}{dt}\left(\frac{m}{\sqrt{1-\frac{l^2 u'(t)^2}{\mathfrak{c}^2}}}\, l\, u'(t)\right),$$

where $\mathfrak{c}$ is the speed of light in the vacuum. Then the resulting equation fits mathematically into the framework of the so-called φ-laplacian operators, [3, 4, 8, 21], and leads to a problem of the form

$$\varphi(u'(t))' + ku'(t) + a\sin u(t) = f(t), \tag{2}$$

with

$$\varphi\colon (-c, c) \to \mathbb{R}, \quad x \mapsto \varphi(x) := \frac{x}{\sqrt{1-\frac{x^2}{c^2}}}, \tag{3}$$

where $c = \frac{\mathfrak{c}}{l}$. However, some papers in the physical literature dealing with relativistic mechanics, like for instance [10, 12], consider that the forces acting on the body depend also on the relativistic mass, which suggests to consider equation

$$\frac{d}{dt}\left(\frac{m}{\sqrt{1-\frac{l^2 u'(t)^2}{\mathfrak{c}^2}}}\, l\, u'(t)\right) = -g\,\frac{m}{\sqrt{1-\frac{l^2 u'(t)^2}{\mathfrak{c}^2}}}\,\sin(u(t)) - \mathfrak{K}\, l\, u'(t) + \mathfrak{f}(t). \tag{4}$$

instead of (2). For a better comparison between both equations it is convenient to rewrite (4) as

$$\varphi(u'(t))' + ku'(t) + a\xi(u'(t))\sin u(t) = f(t), \tag{5}$$

where $k = \frac{\mathfrak{K}}{m}$, $a = \frac{g}{l}$, $f(t) = \frac{\mathfrak{f}(t)}{ml}$, φ is defined by (3) and

$$\xi\colon (-c, c) \to \mathbb{R}, \quad x \mapsto \xi(x) := \frac{1}{\sqrt{1-\frac{x^2}{c^2}}}. \tag{6}$$

Now the difference is clear: the bounded nonlinearity "$a \sin u(t)$" in equation (2) is replaced in (5) by the derivative dependent term "$a\xi(u'(t)) \sin u(t)$", which is unbounded. This apparently slight change introduces a new kind of mathematical complexity in equation (5) since this new term could be unbounded even when one assumes the relativistic restriction $|u'(t)| < c$.

Notice also that if $c \to \infty$ then

$$\varphi(x) \to x \quad \text{and} \quad \xi(x) \to 1,$$

so both (2) and (5) exhibit the right limiting behaviour (that is, both reduce to the classical pendulum equation when $c = +\infty$).

To the best of our knowledge, problem (5) has not been previously considered in the mathematical literature, so the main goal of this paper is to explore the relativistic pendulum equation (5) and extend to it, as much as posible, the known properties satisfied by the classical pendulum equation (1), such as $(P1) - (P2) - (P3)$. In this way we will complement and shed light over previous results in the literature.

More concretely, the paper is organized as follows: in Section 2 we present the main tools used in the paper, namely, the lower and upper solutions method for the periodic problem with derivative dependence and the continuation principle for the Leray-Schauder degree. In Section 3 we study the structure of the solvability set, that is, the set of the values $\alpha \in \mathbb{R}$ such that equation (5) with $f(t) = \tilde{f}(t) + \alpha$ has a T-periodic solution, being $\tilde{f}$ a T-periodic continuous function with mean value zero, i.e., $\frac{1}{T}\int_0^T \tilde{f}(s)ds = 0$. A particularly interesting question –whether or not 0 belongs to the solvability set– is addressed in Section 4. Finally, in Section 5 we use the speed of light as a parameter and focus on the behaviour of the solvability set when the speed of light goes to infinity. Under suitable assumptions the solvability set for the relativistic pendulum will approach the classical one.

We will use the following notation: C_T is the set of the continuous functions $f\colon [0,T] \to \mathbb{R}$ such that $f(0) = f(T)$ and $\tilde{C}_T$ the subset of the functions of C_T with mean value zero. For $f \in C_T$ we denote $\overline{f} := \frac{1}{T}\int_0^T f(s)ds$, $\tilde{f} := f - \overline{f}$, $f_{\min} := \min\limits_{t\in[0,T]} f(t)$ and $f_{\max} := \max\limits_{t\in[0,T]} f(t)$.

2. Preliminaries

For the convenience of the reader we collect in this section the main tools needed in the rest of the paper: the lower and upper solutions method for the periodic and derivative dependent case and the global continuation principle for the Leray-Schauder degree.

2.1. **Lower and upper solutions for the periodic problem depending on the derivative.** We present here a readable version of the method of the lower and upper solutions method and Nagumo's condition suitable for our purposes: consider the periodic boundary value problem

$$u''(t) = h(t, u(t), u'(t)), \quad t \in [0, T], \tag{7}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{8}$$

where $T > 0$ and h is continuous. We will say that $\rho \in C^2([0, T])$ is a lower solution if $\rho(0) = \rho(T)$, $\rho'(0) \geq \rho'(T)$ and

$$\rho''(t) \geq h(t, \rho(t), \rho'(t)), \quad t \in [0, T]. \tag{9}$$

An upper solution $\sigma \in C^2([a, b])$ is defined in a similar way by reversing the previous inequalities.

Roughly speaking, lower and upper solutions provide a priori bounds on u but in order to obtain a priori bounds on u' we will need the so called Nagumo's condition: if $\rho \leq \sigma$ define

$$E = \{(t, u, v) \in [0, T] \times \mathbb{R}^2 \colon \rho(t) \leq u \leq \sigma(t)\},$$

and we say that $h \colon E \to \mathbb{R}$ satisfies Nagumo's condition if

$$|h(t, u, v)| \leq \psi(|v|) \quad \text{for all } (t, u, v) \in E,$$

for some $\psi \colon [0, +\infty) \to \mathbb{R}$ that satisfies

$$\int_0^{+\infty} \frac{s}{\psi(s)} ds = +\infty.$$

Note that, in particular, if h has a subquadratic growth

$$|h(t, u, v)| \leq A + Bv^2 \quad \text{for all } (t, u, v) \in E \text{ and some } A,\ B > 0,$$

then h satisfies Nagumo's condition with $\psi(v) = A + Bv^2$.

The significance of the Nagumo condition is that provides a constant $M > 0$ such any possible solution u of (7)–(8) in E must satisfy $|u'(t)| < M$ for all $t \in [0, T]$.

Theorem 1. *Let ρ and σ lower and upper solutions such that $\rho \leq \sigma$ and suppose that $h \colon E \to \mathbb{R}$ is continuous and satisfies Nagumo's condition.*

Then the problem (7)-(8) *has at least one solution $u \in C^2([0, T])$ such that*

$$\rho(t) \leq u(t) \leq \sigma(t) \quad \textit{for all } t \in [0, T].$$

The interested reader is referred to [14] for an example of nonexistence of periodic solutions in the presence of lower and upper solutions when Nagumo's condition fails and to [9] for a nice overview of the lower and upper solution method.

2.2. **The global continuation principle for the Leray-Schauder degree.** Given a bounded open set $\Omega \neq \emptyset$ in the real Banach space X and $T\colon \overline{\Omega} \subset X \to X$ a compact map such that $Tx \neq x$ for all $x \in \partial\Omega$, the Leray-Schauder degree $deg(I-T,\Omega,0)$ is an integer number that, roughly speaking, counts "algebraically" the number of zeroes of $I-T$ in Ω.

Besides some of its well-known features, such as the homotopy invariance, the existence or the addition–excision properties, the following global continuation principle can be found in [23, Section 14.C] or in [13, IV.13-3.7].

Theorem 2. *Let the operator $T\colon [\mu_1,\mu_2] \times \overline{\Omega} \subset \mathbb{R} \times X \to X$ be compact, where Ω is a bounded open set in the real Banach space X, and consider the parameter dependent equation*

$$T(\mu,x) = x, \quad \mu \in [\mu_1,\mu_2],\, x \in \Omega. \tag{10}$$

If the following assumptions are satisfied

(i) $T(\mu,x) \neq x, \quad \mu \in [\mu_1,\mu_2], \quad x \in \partial\Omega,$
(ii) $deg(I - T(\mu_1,\cdot),\Omega,0) \neq 0,$

then equation (10) *has a continuum of solutions $\mathfrak{S}$, i.e. a compact and connected set in $[\mu_1,\mu_2] \times \overline{\Omega}$, which intersects $\{\mu\} \times \Omega$ for each $\mu \in [\mu_1,\mu_2]$.*

In particular, equation (10) *has a solution in Ω for each $\mu \in [\mu_1,\mu_2]$ and $\mathfrak{C}$ connects $\{\mu_1\} \times \Omega$ with $\{\mu_2\} \times \Omega$.*

We stress that the existence of a continuum of solutions is a powerful consequence of the Leray-Schauder degree that has many interesting consequences, [1]. A connection of the degree with the method of upper and lower solutions is given by the following result, see [9, Chapter III].

Lemma 1. *In the context of Theorem 1, for fixed $\mu > 0$ consider the compact operator $K\colon C^1_T \to C^1_T$ given by $Kg := u$, where u is the unique solution of the linear problem*

$$u''(t) - \mu u(t) = h(t,g(t),g'(t)) - \mu g(t) \tag{11}$$

satisfying the periodic boundary conditions (8).

If $\rho < \sigma$ are strict lower and upper solutions, i.e., the differential inequality (9) *is strict and the reversed strict inequality holds for the upper solution σ, and there exists an a priori bound $M > 0$ for the derivative of any possible $T-$periodic solution of* (11) *enclosed by ρ and σ, then*

$$deg(I-K,\Omega,0) = 1,$$

where

$$\Omega := \{u \in C^1_T\colon \rho(t) < u(t) < \sigma(t),\ |u'(t)| < M,\ t \in [0,T]\}.$$

Remark 1. *As mentioned earlier, the existence of $M > 0$ as in the previous lemma is ensured when the nonlinearity h has subquadratic growth with respect to u' or, more generally, when a Nagumo condition is satisfied.*

3. The structure of the solvability set

Our goal in this section is to study the set $I(a, c, k, T, \tilde{f})$ of the values $\alpha \in \mathbb{R}$ such that equation

$$\varphi(u'(t))' + ku'(t) + a\xi(u'(t))\sin u(t) = \tilde{f}(t) + \alpha, \tag{12}$$

has a T-periodic solution, when $a, c, T > 0$, $k \geq 0$ and $\tilde{f} \in \tilde{C}_T$ is a T-periodic continuous function with mean value zero. Note that we can rewrite (12) equivalently as

$$u''(t) + a\left(1 - \frac{u'(t)^2}{c^2}\right)_+ \sin u(t) = \left(1 - \frac{u'(t)^2}{c^2}\right)_+^{3/2} [-ku'(t) + \tilde{f}(t) + \alpha], \tag{13}$$

provided that $\|u'\|_\infty < c$.

As a first easy result we show that $I(a, c, k, T, \tilde{f})$ is bounded independently on c or k.

Lemma 2. $I(a, c, k, T, \tilde{f}) \subset [-a - \tilde{f}_{\max}, a - \tilde{f}_{\min}]$.

In particular, if $\alpha \in I(a, c, k, T, \tilde{f})$ then $|\alpha| \leq |a| + \|\tilde{f}\|_\infty$.

Proof. If u is a T-periodic solution of (12) for some α, then it attains a maximum at some t_0 where $u'(t_0) = 0 \geq u''(t_0)$, and therefore by taking into account the equivalent formulation (13)

$$0 \geq u''(t_0) = \tilde{f}(t_0) + \alpha - a \sin u(t_0) \geq \tilde{f}_{\min} + \alpha - a.$$

Thus

$$\alpha \leq a - \tilde{f}_{\min}.$$

In an analogous way, u also attains a minimum at some t_1 where

$$0 \leq u''(t_1) = \tilde{f}(t_1) + \alpha - a \sin u(t_1) \leq \tilde{f}_{\max} + \alpha + a.$$

Consequently,

$$\alpha \geq -(\tilde{f}_{\max} + a).$$

and therefore

$$I(a, c, k, T, \tilde{f}) \subset [-a - \tilde{f}_{\max}, a - \tilde{f}_{\min}].$$

□

Also notice that if u is a T-periodic solution of (12), then just integrating the equation over $[0,T]$ it is seen that

$$\alpha = \frac{a}{T}\int_0^T \xi(u'(t))\sin u(t)\,dt.$$

Now, we define the set

$$\mathcal{C} := \{u \in C^1[0,T] : \|u'\|_\infty < c\}$$

and the continuous operator $A : \mathcal{C} \to \mathbb{R}$ given by the right hand side of the previous equation, that is,

$$A(u) := \frac{a}{T}\int_0^T \xi(u'(t))\sin u(t)\,dt.$$

A key result for us will be the close relationship between the periodic and the following integro–differential Dirichlet problem.

Lemma 3. *u is a T-periodic solution of* (12) *for some $\alpha \in \mathbb{R}$ if and only if $u \in \mathcal{C}$, $Au = \alpha$ and there exists $r \in \mathbb{R}$ such that*

$$u''(t) + a\left(1 - \frac{u'(t)^2}{c^2}\right)_+ \sin u(t) = \left(1 - \frac{u'(t)^2}{c^2}\right)_+^{3/2} [-ku'(t) + \tilde{f}(t) + A(u)], \tag{14}$$

$$u(0) = u(T) = r. \tag{15}$$

Consequently, if $U_r \subset \mathcal{C}$ is the set of solutions of (14)-(15)*, then we have*

$$I(a,c,k,T,\tilde{f}) = A\left(\bigcup_{r\in\mathbb{R}} U_r\right) = A\left(\bigcup_{r\in[0,2\pi]} U_r\right) \subset [-a - \tilde{f}_{\max}, a - \tilde{f}_{\min}]. \tag{16}$$

Proof. The sufficiency is clear by the previous considerations. Let us now suppose that $u \in \mathcal{C}$ is a solution of (14)–(15) for some $r > 0$. Since $\|u'\|_\infty < c$, equation (14) is equivalent to

$$\varphi(u'(t))' + ku'(t) + a\xi(u'(t))\sin u(t) = \tilde{f}(t) + A(u), \tag{17}$$

and then integrating (17) on $[0,T]$ we have

$$\varphi(u'(T)) - \varphi(u'(0)) + ku'(T) - ku'(0) = TA(u) - a\int_0^T \xi(u'(t))\sin u(t)\,dt = 0,$$

that is

$$u'(0) = u'(T),$$

since $\varphi(x) + kx$ is a homeomorphism from $(-c,c)$ onto $\mathbb{R}$. Therefore, u can be extended to a T-periodic C^1 function that satisfies (12) for $\alpha = A(u)$. □

Next, we provide another key compactness result.

Lemma 4. *For each $\lambda \in [0,1]$ and $r \in [0,2\pi]$ let $U_r^\lambda \subset \mathcal{C}$ denote the set of solutions of the homotopic family of problems*

$$u''(t)+\lambda a\left(1-\frac{u'(t)^2}{c^2}\right)_+ \sin u(t) = \lambda\left(1-\frac{u'(t)^2}{c^2}\right)_+^{3/2}[-ku'(t)+\tilde{f}(t)+A(u)], \tag{18}$$

subject to the boundary condition (15)*. Then, the set*

$$X := \bigcup_{\lambda\in[0,1]}\bigcup_{r\in[0,2\pi]} U_r^\lambda \tag{19}$$

is a compact subset of $\mathcal{C}$.

Furthermore, for each fixed $\lambda_* \in [0,1]$ and $r_* \in [0,2\pi]$ the sets $\bigcup_{r\in[0,2\pi]} U_r^{\lambda_*}$, $\bigcup_{\lambda\in[0,1]} U_{r^*}^{\lambda}$ and $U_{r_*}^{\lambda_*}$ are also compact.

Proof. We are going to show that each sequence $\{u_n\} \subset X$ has a convergent subsequence in X (which means that X is compact) in such a way that if $u_n \in U_{r_n}^{\lambda_n}$ then it has in fact a subsequence converging to an element of $U_{r_*}^{\lambda_*}$ for some $\lambda_* \in \overline{\{\lambda_n\}}$ and $r_* \in \overline{\{r_n\}}$ (which means in particular that $\bigcup_{r\in[0,2\pi]} U_r^{\lambda_*}$, $\bigcup_{\lambda\in[0,1]} U_{r^*}^{\lambda}$ and $U_{r_*}^{\lambda_*}$ are closed in X and in consequence also compact).

Indeed, let $u_n \in U_{r_n}^{\lambda_n}$ be a solution of (18)-(15) with $\lambda = \lambda_n \in [0,1]$, $r = r_n \in [0,2\pi]$. Then $A(u_n) = \alpha_n \in [-a-\tilde{f}_{\max}, a-\tilde{f}_{\min}]$ because the same bounds that in Lemma 3 are valid independently of $\lambda \in (0,1]$ and for $\lambda = 0$ the only solution is $u \equiv r$ so $|A(u)| \le a$.

Now, since we have for each $n \in \mathbb{N}$

$$\|u_n'\|_\infty < c, \|u_n\|_\infty < 2\pi + cT \quad \text{and} \quad \|u_n''\|_\infty < M,$$

for some M independent of $n \in \mathbb{N}$, by Arzelá-Ascoli there exists a subsequence $\{u_{n_k}\}$ that converges in $C^1[0,T]$ to some function u_* and we can also suppose (passing again to subsequences, if needed) that $\lambda_{n_k} \to \lambda_* \in [0,1]$, $r_{n_k} \to r_* \in [0,2\pi]$ and $\alpha_{n_k} \to \alpha_*$. It follows that u_*' is differentiable, $\|u_*'\|_\infty \le c$, and

$$u_*''(t) + \lambda_* a\left(1-\frac{u_*'(t)^2}{c^2}\right)_+ \sin u_*(t) = \lambda_*\left(1-\frac{u_*'(t)^2}{c^2}\right)_+^{3/2}[-ku_*'(t)+\tilde{f}(t)+\alpha_*], \tag{20}$$

$$u_*(0) = u_*(T) = r_*. \tag{21}$$

Now, we can exclude that $\|u_*'\|_\infty = c$, because in that case, since each straight line with slope $\pm c$ is a solution of equation (20) and this equation satisfies the standard Lipschitz uniqueness condition for the initial value problem, then u_* should be one of such straight lines, which contradicts (21).

Therefore, $u_* \in \mathcal{C}$ and from the continuity of A it follows that $\lim_{k\to\infty} A(u_{n_k}) = A(u_*)$. Then $\alpha_* = A(u_*)$ and, in view of (20)-(21), we have that $u_* \in U_{r_*}^{\lambda *} \subset X$ as we wanted to proof. $\square$

Notice that the compactness result in Lemma 4 does not avoid the fact that X could be empty! Fortunately this is not the case: being X compact means that it is a priori bounded which, together with the homotopy invariance of the Leray-Schauder degree, provides the following existence result.

Lemma 5. *For each* $(r,\lambda) \in [0,2\pi] \times [0,1]$ *it holds that*

$$U_r^\lambda \neq \emptyset.$$

Furthermore, for given $(r_i,\lambda_i) \in [0,2\pi] \times [0,1]$, $i = 1,2$, *there exists a continuum, i.e. a compact and connected set,* $\mathfrak{C} \subset X$, *where* X *is given by* (19) , *such that*

$$\mathfrak{C} \cap U_{r_i}^{\lambda_i} \neq \emptyset.$$

In particular, since $X \subset \mathcal{C}$ *is compact there exist* $R > r$ *and* $0 < c_0 < c$ *such that*

$$\bigcup_{\lambda\in[0,1]} \bigcup_{r\in[0,2\pi]} U_r^\lambda \subset \Omega := \{u \in C^1[0,T] : \|u\|_\infty < R,\ \|u'\|_\infty < c_0\}.$$

Proof. For fixed $r \in [0,2\pi]$, given that $X \subset \mathcal{C}$ is bounded, the homotopy invariance of the Leray-Schauder degree for the associated fixed point operator of problem (18) on Ω ensures that it coincides with that of $I - u_r$, where u_r is the constant function $u_r \equiv r$, that is clearly equal to 1, so $U_r^\lambda \neq \emptyset$ for each $\lambda \in [0,1]$.

Now, the existence of the continuum $\mathfrak{C}$ is a consequence of the global continuation principle of the Leray-Schauder degree (Theorem 2). $\square$

As a consequence of Lemmata 3, 4 and 5 we are able to prove that the solvability set $I(a,c,k,T,\tilde{f})$ is a nonempty compact set and therefore equation (12) always has a T-periodic solution for some $\alpha \in \mathbb{R}$.

Theorem 3. $I(a,c,k,T,\tilde{f}) \neq \emptyset$ *is compact.*

Proof. It is immediate from the fact that $I(a,c,k,T,\tilde{f}) = A\left(\bigcup_{r\in[0,2\pi]} U_r\right)$, the compactness of $\bigcup_{r\in[0,2\pi]} U_r \neq \emptyset$ and the continuity of A. $\square$

In fact, we can now give a much more detailed account on the structure of the solvability set which mimics the Newtonian case (compare for instance with properties $(P1)-(P2)-(P3)$ in the Introduction).

Theorem 4. *For fixed* $a, c, T > 0$, $k \geq 0$ *and given* $\tilde{f} \in \tilde{C}_T$ *let*

$$d(\tilde{f}) := \min I(a, c, k, T, \tilde{f}) \geq -a - \tilde{f}_{max}$$

and

$$D(\tilde{f}) := \max I(a, c, k, T, \tilde{f}) \leq a - \tilde{f}_{min}.$$

Then the following claims hold:

(1) $I(a, c, k, T, \tilde{f}) = [d(\tilde{f}), D(\tilde{f})]$.
(2) $d, D \colon \tilde{C}_T \to \mathbb{R}$ *are continuous functionals.*
(3) *If* $d(\tilde{f}) < D(\tilde{f})$ *and* $\alpha_0 \in (d(\tilde{f}), D(\tilde{f}))$ *then equation* (12) *with* $\alpha = \alpha_0$ *has at least two geometrically different* T*-periodic solutions.*

Proof. Claim 1.- $I(a, c, k, T, \tilde{f})$ *is an interval.*

If $\alpha_1, \alpha_2 \in I(a, c, k, T, \tilde{f})$ and $\alpha_1 < \alpha_* < \alpha_2$ let u_1 and u_2 be associated T-periodic solutions of equation (12) with $\alpha = \alpha_1$ and $\alpha = \alpha_2$. Then u_1 and u_2 are upper and lower solutions, respectively, for the periodic problem associated to (13) with $\alpha = \alpha_*$ which satisfies a Nagumo condition because the nonlinear part of the equation has subquadratic growth in u'. Hence

$$u_2 \leq u_1 + 2k\pi \quad \text{for some } k \in \mathbb{N},$$

and there exists a T-periodic solution for (12) with $\alpha = \alpha_*$ between the well–ordered lower solution u_2 and the upper solution $u_1 + 2k\pi$, that is, $\alpha_* \in I(a, c, k, T, \tilde{f})$.

Claim 2.- $d(\tilde{f})$ *and* $D(\tilde{f})$ *are continuous.*

We are going to prove that if $\{\tilde{f}_n\} \subset \tilde{C}_T$ converges to $\tilde{f}_*$ in $\tilde{C}_T$ then $d(\tilde{f}_n) \to d(\tilde{f}_*)$ (the case $D(\tilde{f}_n) \to D(\tilde{f}_*)$ is analogous).

Firstly, let $\{d(\tilde{f}_n)\}$ denote again a subsequence converging to some value d_*. As in the proof of Lemma 4, we have that if u_n are T-periodic solutions of (13) with $\tilde{f} = \tilde{f}_n$, $\alpha = d(\tilde{f}_n) = A(u_n)$ and $u_n(0) = r_n \in [0, 2\pi]$ then $\{u_n\}$ converges (passing to a subsequence if needed) to a solution u_* of (14) with $\tilde{f} = \tilde{f}_*$, $\alpha = d_* = A(u_*)$ and $r = r_* \in \overline{\{r_n\}}$, which means that $d(\tilde{f}_*) \leq d_*$ and hence

$$d(\tilde{f}_*) \leq \liminf d(\tilde{f}_n).$$

So, it suffices to prove that $\limsup d(\tilde{f}_n) \leq d(\tilde{f}_*)$. By contradiction, suppose that $d(\tilde{f}_*) < \limsup d(\tilde{f}_n))$, then there exists s such that for some large enough n we have

$$d(\tilde{f}_*) < s < d(\tilde{f}_n) \quad \text{and} \quad \tilde{f}_* + d(\tilde{f}_*) < \tilde{f}_n + s < \tilde{f}_n + d(\tilde{f}_n).$$

Let u_* and u_n denote T-periodic solutions of (13) with $\tilde{f} = \tilde{f}_*$ and $\alpha = d(\tilde{f}_*)$ and with $\tilde{f} = \tilde{f}_n$ and $\alpha = d(\tilde{f}_n)$, respectively. In this case, u_n and u_* are lower and upper solutions, respectively, for (13) with $\tilde{f} = \tilde{f}_n$ and $\alpha = s$, which implies that $s \in I(a, c, k, T, \tilde{f}_n)$. This is a contradiction since by construction $s < d(\tilde{f}_n) = \min I(a, c, k, T, \tilde{f}_n)$.

Claim 3.- If $d(\tilde{f}) < \alpha_0 < D(\tilde{f})$ then equation (13) *with $\alpha = \alpha_0$ has at least two geometrically different T-periodic solutions.*

Let u_1 and u_2 solutions for (13) with $\alpha = d(\tilde{f})$ and $\alpha = D(\tilde{f})$, respectively. Then, u_1 and u_2 are strict upper and lower solutions for (13) with $\alpha = \alpha_0$, respectively. By the 2π-periodicity of (13), we can suppose that

$$u_2 < u_1 \quad \text{and} \quad u_2 + 2\pi \not< u_1.$$

In this way, if we consider for $j = 0, 1$

$$\Omega_j := \{u : u_2 + 2j\pi < u < u_1 + 2j\pi,\ \|u'\|_\infty < c_0|\},$$

which are both subsets of

$$\Omega := \{u : u_2 < u < u_1 + 2\pi,\ \|u'\|_\infty < c_0\},$$

where c_0 is given in Lemma 5. Then, from Lemma 1, we may fix $\mu = 1$ and the associated operator $I - K$ has Leray–Schauder degree equal to 1 in the three sets. Since $\overline{\Omega_0} \cap \overline{\Omega_1} = \emptyset$ the addition–excision property of the Leray–Schauder degree implies that the degree of the associated operator in the open set $\Omega \setminus \left(\overline{\Omega_0} \cup \overline{\Omega_1}\right)$ is -1 and therefore there exists T-periodic solutions for (13) with $\alpha = \alpha_0$ in Ω_0 and Ω which cannot differ by a multiple of 2π. □

Remark 2. *As d and D are continuous the strict inequality $d(\tilde{f}) < D(\tilde{f})$ holds in an open set of $\tilde{C}_T$. The case $d(\tilde{f}) = D(\tilde{f})$, in which case the interval $I(a, c, k, T, \tilde{f})$ reduces to a point, is called "degenerate" and it is an open problem, even in the Newtonian case, to prove or disprove if such situation is possible for some $\tilde{f}$.*

Notice that if the problem is degenerate, that is, if $I(a, c, k, T, \tilde{f}) = \{s\}$, employing Lemma 5 it is possible to prove that the set of all T-periodic solutions of (13) *with $\alpha = s$ is homeomorphic to the real line. In more precise terms, for each $r \in \mathbb{R}$, the set U_r consists in a unique element u_r and the map $r \mapsto u_r$ is continuous (and hence a homeomorphism).*

For a more detailed discussion on equivalent conditions to degeneracy in the Newtonian case, see [19].

4. When does zero belong to the solvability set?

This is an interesting question whose answer for the classical pendulum equation is always affirmative in case $k = 0$, [16], whereas in case $k > 0$ fails quite spectacularly, [18]: for any $k > 0$, $a > 0$ and $T > 0$ there exists $\tilde{f} \in \tilde{C}_T$ such that equation

$$u''(t) + ku'(t) + a \sin u(t) = \tilde{f}(t), \tag{22}$$

does not have any T-periodic solution.

In our setting the last scenario is avoided for small values of the period T as we will see in the following result, precisely, when $cT \leq \sqrt{3}\pi$. This fact is well–known for equation (2), see [3, 20], and in that context even better estimates were obtained, see [2, 7]. However we do not know how to improve the bound $\sqrt{3}\pi$ when dealing with equation (5).

Theorem 5. *If $cT \leq \sqrt{3}\pi$ then for any $a > 0$, $k \geq 0$ and $\tilde{f} \in \tilde{C}_T$ holds that*

$$0 \in I(a, c, k, T, \tilde{f})^\circ,$$

and in particular there exist at least two geometrically different T-periodic solutions for equation (12) *with $\alpha = 0$.*

Proof. It is enough to use the continuation method in the set

$$\Omega := \{u \in C^1_T : |\overline{u}| < \frac{\pi}{2}, \|u'\|_\infty < c_0\}.$$

by showing that the family

$$\varphi(u'(t))' = \lambda[\tilde{f}(t) - a\xi(u'(t)) \sin u(t)], \tag{23}$$

does not have solutions in $\partial\Omega$ for any $\lambda \in [0, 1]$.

Indeed, if u satisfies (23) then $\|u'\|_\infty < c_0 < c$ and therefore $u \in \partial\Omega$ implies that $\overline{u} = \frac{\pi}{2}$ (or maybe $\overline{u} = -\frac{\pi}{2}$, but this case is analogous). Now, taking into account the Sobolev inequality for $u \in C^1_T$, namely,

$$\|u - \overline{u}\|_\infty \leq \frac{T}{2\sqrt{3}} \|u'\|_\infty,$$

and we have that

$$\left\| u - \frac{\pi}{2} \right\|_\infty < \frac{c_0}{c} \frac{\pi}{2} < \frac{\pi}{2}.$$

Thus

$$0 < u(t) < \pi \quad \text{for all } t \in [0, T],$$

which means that $\sin u(t)$ does not vanish. On the other hand, integrating (23) on $[0,T]$ and using the T-periodicity we obtain

$$0 = \int_0^T \xi(u'(t)) \sin u(t)\, dt,$$

a contradiction.

In fact, the same reasoning shows that if $\varepsilon > 0$ is small enough then equation (23) with $\tilde{f} \pm \varepsilon$ still has a T-periodic solution and then

$$(-\varepsilon, \varepsilon) \subset I(a, c, k, T, \tilde{f}),$$

as we wanted to show. □

Remark 3. *To the best of our knowledge the following is in general an open question: does $0 \in I(a, c, 0, T, \tilde{f})$ for any $a > 0$, $c > 0$, $T > 0$ and $\tilde{f} \in \tilde{C}_T$?*

Notice that the proof of this fact, for the classical pendulum or for equation (2) *with $k = 0$, is variational and we do not know if the result can be extended to our setting which does not have a variational structure even if $k = 0$. We are only able to provide an affirmative answer in some specific situations: for instance, this is obvious when $\|\tilde{f}\|_\infty \le a$, because in this case we may take $\rho = \frac{\pi}{2}$ and $\sigma = \frac{3\pi}{2}$ as an ordered couple of lower and upper solutions for $\alpha = 0$. Another easy example is shown in the following result.*

Proposition 1. *For any $a > 0$, $c > 0$, $T > 0$ and $\tilde{f} \in \tilde{C}_T$ odd we have that*

$$0 \in I(a, c, 0, T, \tilde{f}).$$

Proof. We know by Theorem 4 that $I(a, c, 0, T, \tilde{f})$ is a nonempty interval and moreover it is symmetric with respect to the origin, since if u is a solution for $\tilde{f} + \alpha$ then $v(t) = -u(-t)$ is a solution for $\tilde{f} - \alpha$. So, $0 \in I(a, c, 0, T, \tilde{f})$. □

5. The dependence of I on c

Remember that the parameter $c = \dfrac{\mathfrak{c}}{l}$, so it is proportional directly to the speed of light $\mathfrak{c}$. The Newtonian (or classical dynamics) corresponds to the limiting case $c = +\infty$ and, although Einstein's special relativity prescribes a fixed and finite value for $\mathfrak{c}$, from the mathematical viewpoint it is both interesting and fruitful trying to continuate the solutions of the classical pendulum equation to the relativistic setting for large enough values of c.

To this end, let $\varepsilon := \frac{1}{c^2}$ and write equation (13) as

$$u''(t) + a(1 - \varepsilon u'(t)^2)_+ \sin u(t) = (1 - \varepsilon u'(t)^2)_+^{3/2}[-ku'(t) + \tilde{f}(t) + \alpha]. \tag{24}$$

Of course, we always mean that $\varepsilon u'(t)^2 < 1$.

Lemma 6. *If the classical pendulum equation (that is, equation* (24) *with* $\varepsilon = 0$*) admits a pair of strict lower and upper solutions then there exists* $c^* > 0$ *such that the relativistic pendulum equation* (13) *has a* T*–periodic solution for all* $c \geq c^*$*.*

Proof. When $\varepsilon = 0$, if u is T-periodic, then integration at both sides yields

$$\frac{a}{T}\int_0^T \sin u(t)dt = \alpha,$$

that is, $|\alpha| \leq a$. This implies the existence of a constant $M > 0$ such that $\|u'\|_\infty < M$ for all possible T-periodic solutions. Indeed, when $k = 0$ simply observe that

$$\|u''\|_{L^2} \leq a\|\sin u\|_{L^2} + \|\tilde{f} + \alpha\|_{L^2} \leq 2aT^{1/2} + \|\tilde{f}\|_{L^2}$$

and the conclusion follows from the fact that $\|u'\|_\infty \leq T^{1/2}\|u''\|_{L^2}$. On the other hand, for $k > 0$ multiply the equation by u' and integrate to obtain

$$k\|u'\|_{L^2}^2 = \int_0^T (\tilde{f}(t) + \alpha)u'(t)dt,$$

whence $k\|u'\|_{L^2} \leq \|\tilde{f} + \alpha\|_{L^2}$. This, in turn, implies $\|u''\|_{L^2} \leq 3aT^{1/2} + \|\tilde{f}\|_{L^2}$ and the proof follows as before.

Next, assume that $u_*, u^* \in C_T^2$ are respectively strict lower and upper solutions of (24) for $\varepsilon = 0$ such that $u_*(t) < u^*(t)$ for all t. Consider the set

$$\Omega := \{u \in C_T^1 \colon u_*(t) < u(t) < u^*(t),\ |u'(t)| < M, \quad \forall\, t \in \mathbb{R}\}$$

and the compact operator K_ε defined as in Lemma 1 with $\mu = 1$ and

$$h_\varepsilon(t,u,v) := -a(1-\varepsilon v^2)_+ \sin u + (1-\varepsilon v^2)_+^{3/2}[-kv + \tilde{f}(t) + \alpha].$$

Therefore Lemma 1 implies that $\deg(I - K_0, \Omega, 0) = 1$. Now we claim that, when ε is sufficiently small, $K_\varepsilon(u) \neq u$ for $u \in \partial\Omega$ and, consequently, $\deg(I - K_\varepsilon, \Omega, 0) = 1$. Indeed, suppose that $K(u_n, \varepsilon_n) = u_n$ for some $u_n \in \partial\Omega$ and $\varepsilon_n \to 0$. Passing to a subsequence, we may suppose that $u_n \to u$ for some $u \in \partial\Omega$, whence $K_0(u) = u$, a contradiction. □

According to the preceding result, we may denote by $I(a, \infty, k, T, \tilde{f})$ the solvability set for the classical pendulum equation, that is, for $\varepsilon = 0$. As already mentioned, the claims established in Theorem 4 are valid for this case, with $-a \leq d(\tilde{f}) \leq D(\tilde{f}) \leq a$. For convenience, we shall add a sub-index to the quantities d and D in order to emphasize the dependence on $c \in (0, +\infty]$. In particular, when $d_\infty(\tilde{f}) < \alpha < D_\infty(\tilde{f})$, the existence of two geometrically different solutions for $c \gg 0$ follows from the previous considerations. Furthermore, we may establish the following:

Theorem 6. *Assume that* $J := [d, D] \subset I(a, \infty, k, T, \tilde{f})^\circ$. *Then there exists* c_J *such that* $J \subset I(a, c, k, T, \tilde{f})$ *for all* $c > c_J$. *In particular,*

$$\limsup_{c\to\infty} d_c(\tilde{f}) \le d_\infty(\tilde{f}), \qquad D_\infty(\tilde{f}) \le \liminf_{c\to\infty} D_c(\tilde{f}).$$

Proof. From the previous considerations, there exists ε_d such that the problem has (at least two) solutions when $\varepsilon < \varepsilon_d$, and the same is true for D. Thus, it suffices to set $\varepsilon_J := \min\{\varepsilon_d, \varepsilon_D\}$ and $c_J := \frac{1}{\sqrt{\varepsilon_J}}$.

□

Now we prove that the derivative of the T-periodic solutions is uniformly bounded with respect to c.

Lemma 7. *There exists* $M > 0$ *(independent of* ε*) such that any* T*-periodic solution* u_ε *of (24) such that* $\|u_\varepsilon'\|_\infty < \frac{1}{\sqrt{\varepsilon}}$ *satisfies* $\|u_\varepsilon'\|_\infty < M$ *for any* $\varepsilon > 0$.

Proof. By multiplying (24) by $\frac{u_\varepsilon''(t)}{1-\varepsilon u_\varepsilon'(t)^2}$ and integrating we obtain

$$\begin{aligned} \|u_\varepsilon''\|_{L^2}^2 &\le \int_0^T \frac{u_\varepsilon''(t)^2}{1-\varepsilon u_\varepsilon'(t)^2} dt \\ &= \int_0^T (1-\varepsilon u_\varepsilon'(t)^2)^{1/2}(\tilde{f}(t)+\alpha)u_\varepsilon''(t)dt - a\int_0^T \sin u_\varepsilon(t) u_\varepsilon''(t)dt \\ &\le \|\tilde{f}+\alpha\|_{L^2}\|u_\varepsilon''\|_{L^2} + aT^{1/2}\|u_\varepsilon''\|_{L^2}. \end{aligned}$$

We conclude from Lemma 2 that $\|u_\varepsilon''\|_{L^2} \le C$ for some constant C independent of ε and the proof follows. □

As a consequence, we obtain the following nonexistence result that resembles in our framework that of Torres [22].

Theorem 7. *We have,*

$$d_\infty(\tilde{f}) \le \liminf_{c\to\infty} d_c(\tilde{f}), \qquad \limsup_{c\to\infty} D_c(\tilde{f}) \le D_\infty(\tilde{f}).$$

In particular, if $\alpha \notin I(a, \infty, k, T, \tilde{f})$ *then there exists* $c^* > 0$ *such that* $\alpha \notin I(a, c, k, T, \tilde{f})$ *for all* $c > c^*$.

Proof. Let $\{u_n\}$ be a sequence of T-periodic solutions for some $c_n \to \infty$ and $\alpha_n = D_{c_n}(\tilde{f})$ converging to some α. Set $\varepsilon_n := \frac{1}{c_n^2}$ and assume w.l.o.g that $u_n(0) \in [0, 2\pi]$. From the proof of the previous lemma, it follows that $\|u_n''\|_{L^2}$ and $\|u_n'\|_\infty$ are uniformly bounded. From the Arzelà-Ascoli theorem we may assume that u_n converges in C_T to some function u. Passing to the limit in equation (24), it is seen that $u_n'' \to -ku' - a\sin u + \tilde{f} + \alpha$ uniformly. It follows that $u \in C_T^2$ and solves (24) for $\varepsilon = 0$ which, in turn, implies $\alpha \le D_\infty(\tilde{f})$. The proof is analogous for $d_\infty(\tilde{f})$.

□

As a consequence of the previous result we see that condition $cT \leq \sqrt{3}\pi$ cannot be completely dropped out in Theorem 5.

Corollary 1. *Let $a > 0$, $k > 0$ and $T > 0$. Then there exists $\tilde{f} \in \tilde{C}_T$ and $c^* > 0$ such that*

$$0 \notin I(a, c, k, T, \tilde{f}) \quad \textit{for all } c > c^*.$$

Proof. It is known from [18] that given $a > 0$, $k > 0$ and $T > 0$ there exists $\tilde{f} \in \tilde{C}_T$ such that equation (22) does not have any T-periodic solution, that is

$$0 \notin I(a, \infty, k, T, \tilde{f}).$$

Thus, the result follows from Theorem 7. □

Also, as a direct consequence of Theorems 6 and 7, we obtain sufficient conditions for the solvability set $I(a, c, k, T, \tilde{f})$ of the relativistic pendulum to converge when $c \to \infty$ to the classical pendulum solvability set $I(a, \infty, k, T, \tilde{f})$:

Corollary 2. *Suposse that $I(a, \infty, k, T, \tilde{f})^\circ \neq \emptyset$. Then*

$$\lim_{c \to \infty} d_c(\tilde{f}) = d_\infty(\tilde{f}) \quad \textit{and} \quad \lim_{c \to \infty} D_c(\tilde{f}) = D_\infty(\tilde{f}).$$

Remark 4. *Note that some of the previous results could be reinterpreted in terms of the closed parameter set*

$$\Gamma = \Gamma(a, k, T, \tilde{f}) := \{c \in (0, +\infty] \colon 0 \in I(a, c, k, T, \tilde{f})\}.$$

For instance:

- *Theorem 5 implies that Γ always contains a neighborhood of the origin;*
- *Corollary 1 shows that in some cases Γ is bounded;*
- *Corollary 2 implies, in particular, that if $k = 0$ and the corresponding classical pendulum equation is nondegenerate, then Γ contains also a neighborhood of $+\infty$.*

On the other hand, the open question in Remark 3 is equivalent to asking if for $k = 0$ we always have that $\Gamma = (0, +\infty]$.

References

[1] P. Amster, On a theorem by Browder and its application to nonlinear boundary value problems, *Bull. London Math. Soc.* 2023, 1–19.

[2] P. Amster, M. P. Kuna and D. P. Santos, On the solvability of the periodically forced relativistic pendulum equation on time scales, *Electron. J. Qual. Theory Differ. Equ.* **2020**, No. 62, 1–11.

[3] C. Bereanu, P. Jebelean and J. Mawhin, Periodic solutions of pendulum-like perturbations of singular and bounded Φ–Laplacians, *Journal of Dynamics and Differential Equations*, **22** (2010), 463-471.

[4] C. Bereanu and J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, *J. Differential Equations* **Vol. 243**, Issue 2, (2007), 536-557.

[5] C. Bereanu and P. J. Torres, Existence of at least two periodic solutions of the forced relativistic pendulum, *Proc. Am. Math. Soc.* **140** (2012), no. 8, 2713–2719.

[6] H. Brezis and J. Mawhin, Periodic solutions of the forced relativistic pendulum, *Differential Integral Equations* **23**, (2010), 801-810.

[7] J. A. Cid, On the existence of periodic oscillations for pendulum-type equations, *Adv. Nonlinear Anal.* 2021, 10, 121–130.

[8] J. A. Cid and P. J. Torres, On the existence and stability of periodic solutions for pendulum-like equations with friction and ϕ-Laplacian, *Discrete Contin. Dyn. Syst.*, **33** (2013), 141–152.

[9] C. De Coster and P. Habets, *Two-point boundary value problems: lower and upper solutions*, Mathematics in Science and Engineering, Elsevier, 2006.

[10] C. Erkal, The simple pendulum: a relativistic revisit, *Eur. J. Phys.* 21 (2000) 377-384.

[11] H. Goldstein, *Classical Mechanics*, Second Edition, Addison Wesley, (1980).

[12] H. F. Goldstein and C. M. Bender, Relativistic brachistochrone, *J. Math. Phys.* 27 (1986), no. 2, 507–511.

[13] A. Granas and J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, (2003),

[14] P. Habets and R. L. Pouso, Examples of the nonexistence of a solution in the presence of upper and lower solutions, *A*NZIAM J., **44** (2003), 591–594.

[15] J. Mawhin, The forced pendulum equation: a paradigm for nonlinear analysis and dynamical systems, *Expo. Math.* **6** (1988), 271–287.

[16] J. Mawhin, Global results for the forced pendulum equation, in *Handbook of Differential Equations*, Vol. 1 (2004), Elsevier, 533-589.

[17] J. Mawhin and M. Willem, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, *J. Differential Equations*, **52** (1984), 264-287.

[18] R. Ortega, E. Serra and M. Tarallo, Non-continuation of the periodic oscillations of a forced pendulum in the presence of friction, *Proc. Amer. Math. Soc.* **128**, no. 9, (2000), 2659–2665.

[19] R. Ortega and M. Tarallo, Degenerate equations of pendulum–type, *Commun. Contemp. Math.* 2 (2000), no. 2, 127–149.

[20] P. J. Torres, Periodic oscillations of the relativistic pendulum with friction, *Phys. Lett. A* 2008, no. 42, 6386–6387.

[21] P. J. Torres, Nondegeneracy of the periodically forced Liénard differential equation with ϕ-Laplacian, *Communications in Contemporary Mathematics*, Vol. **13**, No. 2 (2011) 283–292.

[22] P. J. Torres, A non–existence result for periodic solutions of the relativistic pendulum with friction, *Appl. Math. Lett.* 144 (2023).

[23] E. Zeidler, *Nonlinear functional analysis and its applications. I. Fixed-point theorems*, Springer-Verlag, New York (1986).

[1]Departamento de Matemática, Facultad de Ciencias y Naturales, Universidad de Buenos Aires – IMAS-CONICET

Email address: pamster@dm.uba.ar

[2]CITMAga, 15782 Santiago de Compostela, Spain

[3]Universidade de Vigo, Departamento de Matemáticas, Campus de Ourense, 32004, Spain

Email address: angelcid@uvigo.es