Solvability of discontinuous functional differential systems in $l_\infty(M)$

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Abstract: We study the existence of extremal solutions for an infinite system of first order discontinuous functional differential equations in the Banach space of the bounded functions $l_\infty(M)$.
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1 Introduction

The study of functional differential equations covers, among others, ordinary differential equations, integro-differential equations and equations with maxima. Moreover this type of equations appear when we use the reduction of order method to a suitable scalar n th order ordinary differential equation, [2, 3], which can be, obviously, treated as a first order system of finite equations. Follow this idea, one can consider differential functional systems with infinity equations, not necessary countable, which have been studied by different authors, see [6] and the references therein.

In this paper we study the solvability of an infinite system of functional differential equations, with nonlinear functional boundary value conditions, in the Banach space of the bounded functions $l_\infty(M)$, where M is an arbitrary set of index,

$$\begin{cases} u'_\nu(t) = g_\nu(t, u(t), u) \text{ for a.a. } t \in I := [t_0, t_1], \qquad \nu \in M, \\ \\ u_\nu(t_0) = B_\nu(u(t_0), u), \qquad \nu \in M. \end{cases}$$

Our main result extends [4, theorem 3.1] to infinite systems and it also improves [1, theorem 1.1] and [7]. The ideas contained in the proof of our main result are related to those of [6].

2 Definitions and Preliminaries

We say that a partially ordered set (*poset*) X is a *lattice* if $\sup\{x_1,\ x_2\}$ and $\inf\{x_1,\ x_2\}$ exist for all $x_1,\ x_2 \in X$. A lattice X is *complete* when each non empty subset $Y \subset X$ has the supremum and the infimum in X. In particular, every complete lattice has the maximum and the minimum.

In a poset X we define for each $a,\, b \in X$, with $a \leq b$, the interval

$$[a,b] := \{x \in X\ :\ a \leq x \leq b\}.$$

The following result is the well-known Tarski's fixed point theorem (see [12]).

THEOREM 2.1 *Every nondecreasing mapping $G : X \to X$ on a complete lattice X has the minimal, x_*, and the maximal fixed point, x^*. Moreover,*

$$x_* = \min\{x \in X\ :\ Gx \leq x\}, \quad x^* = \max\{x \in X\ :\ x \leq Gx\}.$$

Let M be an arbitrary index set. An element $x := (x_\nu)_{\nu \in M}$ of $\mathbb{R}^M$ is denoted by $x := (x_\nu, x^\nu)$ where $x^\nu \in \mathbb{R}^{M\setminus\{\nu\}}$. If $x, y \in \mathbb{R}^M$ we define the partial ordering

$$x \leq y \quad \text{if and only if} \quad x_\nu \leq y_\nu \text{ for all } \nu \in M.$$

We consider the Banach space

$$l_\infty(M) = \{x := (x_\nu)_{\nu \in M} \in \mathbb{R}^M : \|x\| := \sup_{\nu \in M} |x_\nu| < +\infty\},$$

and for the interval $I = [t_0, t_1]$ we define $C(I, l_\infty(M))$ as the Banach space of all continuous functions $u : I \to l_\infty(M)$ with the norm

$$\|u\|_0 = \sup\{\|u(t)\| : t \in I\},$$

and we define the partial ordering, $u \leq v$ if and only if $u(t) \leq v(t)$ for all $t \in I$.

The following fixed point theorem is essentially [8, theorem 4] (see also remark (6.3) in [8]).

THEOREM 2.2 *Let $a, b \in \mathbb{R}^M$, with $a \leq b$, and $f := (f_\nu)_{\nu \in M} : [a,b] \to \mathbb{R}^M$ be a function such that $f(a) \leq a$ and $b \leq f(b)$. Suppose that f satisfies the following properties for each $\nu \in M$ and for each $x \in [a,b]$:*

(i) The function $f_\nu(\cdot, x^\nu)$ *is u.s.c. on the right and l.s.c. on the left on* $[a_\nu, b_\nu]$*, that is,*

$$\limsup_{y\to x_\nu^-} f_\nu(y, x^\nu) \le f_\nu(x_\nu, x^\nu) \le \liminf_{y\to x_\nu^+} f_\nu(y, x^\nu).$$

(ii) The function f *is quasimonotone, that is,* $f_\nu(x_\nu, \cdot)$ *is nondecreasing on* $[a^\nu, b^\nu]$.

Then, the function f *has the minimal,* $x_* \in [a, b]$*, and the maximal,* $x^* \in [a, b]$*, fixed points and moreover they satisfy the properties*

$$x_* = \min\{x \in [a, b] \ : \ f(x) \le x\}, \tag{2.1}$$

$$x^* = \max\{x \in [a, b] \ : \ x \le f(x)\}. \tag{2.2}$$

Remark 2.1 *Theorem 2.2 extends to quasimonotone maps defined in arbitrary product spaces some earlier fixed point theorems by Hu and Schmidt, [9, 11], for quasimonotone maps defined in* $\mathbb{R}^n$ *and sequence spaces, respectively.*

To end this section we introduce the classical concept of lower solution of the scalar initial value problem

$$u'(t) = h(t, u(t)), \quad \text{for a. e. } t \in [t_0, t_1]; \qquad u(t_0) = A,$$

with h a Carathéodory function, as a function $\alpha \in AC([t_0, t_1])$ that satisfies the following inequalities

$$\alpha'(t) \le h(t, \alpha(t)), \quad \text{for a. e. } t \in [t_0, t_1]; \qquad \alpha(t_0) \le A.$$

The concept of upper solution is given by reversing the previous inequalities. A solution of such problem will be a function that is both a lower and an upper solution.

3 Main Result

In this section we study the problem

$$\begin{cases} u_\nu'(t) = g_\nu(t, u(t), u) \text{ for a.a. } t \in I := [t_0, t_1], \qquad \nu \in M, \\ \\ u_\nu(t_0) = B_\nu(u(t_0), u), \qquad \nu \in M, \end{cases} \tag{3.1}$$

assuming that $g := (g_\nu)_{\nu\in M} : I \times l_\infty(M) \times C(I, l_\infty(M)) \to l_\infty(M)$ and $B := (B_\nu)_{\nu\in M} : l_\infty(M) \times C(I, l_\infty(M)) \to l_\infty(M)$ satisfy for each $\nu \in M$ the following list of hypotheses which we will denote by $(\mathcal{A})$:

(g0) For all $u = (u_\nu, u^\nu) \in C(I, l_\infty(M))$ and all $z \in \mathbb{R}$ the function $t \to g_\nu(t, z, u^\nu(t), u)$ is Lebesgue measurable.

(g1) For a.a. $t \in I$ and for all $x = (x_\nu, x^\nu) \in l_\infty(M)$ and $u \in C(I, l_\infty(M))$ the function $g_\nu(t, x_\nu, \cdot, u)$ and

$$\limsup_{y\to x_\nu^-} g_\nu(t, y, x^\nu, u) \le g_\nu(t, x_\nu, x^\nu, u) \le \liminf_{y\to x_\nu^+} g_\nu(t, y, x^\nu, u).$$

(g2) For a.a. $t \in I$ and for all $x \in l_\infty(M)$ the function $g_\nu(t, x, \cdot)$ is nondecreasing.

(g3) There exist $p, q, r \in L^1_+(I)$ such that for a.a. $t \in I$ and for all $x \in l_\infty(M)$ and $u \in C(I, l_\infty(M))$ we have

$$\|g(t, x, u)\| \le p(t)\|x\| + q(t)\|u\|_0 + r(t)$$

(g4) $\|p\|_{L^1} + \|q\|_{L^1} < 1$.

(B0) For each $x \in l_\infty(M)$ the operator $B_\nu(x, \cdot)$ is nondecreasing.

(B1) For all $x = (x_\nu, x^\nu) \in l_\infty(M)$ and $u \in C(I, l_\infty(M))$ the function $B_\nu(x_\nu, \cdot, u)$ is nondecreasing and

$$\limsup_{y\to x_\nu^-} B_\nu(y, x^\nu, u) \le B_\nu(x_\nu, x^\nu, u) \le \liminf_{y\to x_\nu^+} B_\nu(y, x^\nu, u).$$

(B2) There exist $a, b \in l_\infty(M)$, with $a \le b$, such that for all $u \in C(I, l_\infty(M))$

$$a \le B(a, u) \quad \text{and} \quad B(b, u) \le b.$$

Definition 3.1 *We say that $u = (u_\nu)_{\nu\in M} \in C(I, L_\infty(M))$ is a solution of problem (3.1) if it satisfies $u_\nu \in AC(I)$ for all $\nu \in M$ and*

$$\begin{cases} u_\nu'(t) = g_\nu(t, u(t), u) \;\; \textit{for a.a. } t \in I, \quad \nu \in M, \\ \\ u_\nu(t_0) = B_\nu(u(t_0), u), \quad \nu \in M. \end{cases} \tag{3.2}$$

Now we are ready to present our main result.

THEOREM 3.1 *Assume the list of hypotheses* $(\mathcal{A})$. *Then the problem (3.1) has the minimal and the maximal solution in the set*

$$Y = \{u \in C(I, L_\infty(M)) : a \le u(t_0) \le b\}.$$

Proof. We prove the existence of the maximal solution in Y since the existence of the minimal solution is proved by dual arguments.

For each $u \in C(I, l_\infty(M))$ we define the operator

$$N(u) := \text{the maximal fixed point in } [a,b] \text{ of function } B(\cdot, u).$$

Operator N is well defined by hypotheses $(B1)$, $(B2)$ and theorem 2.2. Moreover, from $(B0)$ and (2.2) it follows easily that N is nondecreasing.

Now, we define

$$R := \frac{\max\{\|a\|, \|b\|\} + \|r\|_{L^1}}{1 - (\|p\|_{L^1} + \|q\|_{L^1})},$$

$$h(t) = R[p(t) + q(t)] + r(t),$$

$$C = \left\{ w : I \to \mathbb{R} : w \in [-R, R],\ |w(s) - w(t)| \le \left| \int_s^t h(r)dr \right| \ \forall\, t, s \in I \right\}$$

and $X = \prod_{\nu \in M} C$. Clearly $X \subset C(I, l_\infty(M))$ and we consider for each $\nu \in M$ the operator $G_\nu : X \to C$ defined for each $v = (v_\nu, v^\nu) \in X$ as the maximal solution of the scalar initial value problem

$$\begin{cases} z'(t) = g_\nu^v(t, z(t)) \text{ for a.a. } t \in I, \\ \\ z(t_0) = N_\nu(v), \end{cases} \tag{3.3}$$

where the scalar function $g_\nu^v : I \times \mathbb{R} \to \mathbb{R}$ is defined for all $(t, z) \in I \times \mathbb{R}$ as

$$g_\nu^v(t, z) = g_\nu(t, z, v^\nu(t), v).$$

Claim 1. $G_\nu : X \to C$ *is well defined.*

For each $\nu \in M$ and $v = (v_\nu, v^\nu) \in X$ we consider the functions

$$\beta(t) = (1 + R)\mathrm{e}^{\int_{t_0}^t [p(s) + q(s)R + r(s)]ds} - 1$$

and $\alpha(t) = -\beta(t)$ for all $t \in I$. It is easy to verify that $\beta(t) \geq R \geq -R \geq \alpha(t)$ for all $t \in I$ and that α and β are lower and upper solutions, respectively, for problem (3.3). Moreover, by hypotheses $(g0), (g1)$ and $(g3)$ the function g_ν^v satisfies conditions 1, 2 and 3 of [10, theorem 2.4] and thus there exist the maximal solution, z^*, of problem (3.3) in $[\alpha, \beta]$, which moreover satisfies

$$z^* = \max\{z \in [\alpha, \beta] \,:\, z'(t) \leq g_\nu^v(t, z(t)) \,\text{a.e.}\; I,\; z(t_0) \leq N_\nu(v)\}. \tag{3.4}$$

Furthermore it is easy to check that any solution z of problem (3.3) satisfies that $\|z\| \leq R$ and therefore $z \in [\alpha, \beta]$. Thus z^* is the maximal solution of problem (3.3) (not only in $[\alpha, \beta]$).

Claim 2. X is a complete lattice.

Since $X = \prod_{\nu \in M} C$ it is enough to prove that C is a complete lattice. Given a nonempty subset $Y \subset C$ it is easy to prove that

$$w_*(t) := \inf\{w(t) : w \in Y\} \quad \text{and} \quad w^*(t) := \sup\{w(t) : w \in Y\} \quad \text{for all } t \in I,$$

are the infimum and the supremum of Y in C, respectively.

Claim 3. $G := (G_\nu)_{\nu \in M} : X \to X$ is nondecreasing.

By using hypotheses $(g1), (g2)$, the fact that N is nondecreasing and property (3.4), it is easy to prove that $G_\nu : X \to C$ is nondecreasing for all $\nu \in M$.

By *Claims* 2 and 3, Tarski's fixed point theorem ensures that G has the maximal fixed point, $u^* \in X$, which satisfies

$$u^* = \max\{u \in X \;:\; u \leq Gu\} \tag{3.5}$$

Claim 4. The maximal fixed point of G, u^*, is the maximal solution in Y of problem (3.1).

Clearly, u^* is a solution in Y of problem (3.1). Let u be another solution in Y of (3.1). Then it is easy to verify that $u \in X$ and $u \leq Gu$. Therefore from (3.5) it follows that $u \leq u^*$ and thus u^* is the maximal solution of problem (3.1). $\square$

Corollary 3.1 *Assume hypotheses (g0)–(g4), (B0), (B1) and*

$(\overline{B2})$

$$\limsup_{\|x\|\to\infty} \frac{\|B(x,u)\|}{\|x\|} < 1, \qquad \textit{uniformly at} \quad u \in C(I, l_\infty(M)).$$

Then the problem (3.1) has the minimal and the maximal solution.

Proof. Let

$$\limsup_{\|x\|\to\infty} \frac{\|B(x,u)\|}{\|x\|} = c < 1.$$

By choosing in the definition of lim sup the value of $\epsilon = (1-c)/2 > 0$, we have that there exists $K > 0$ such that for all $u \in C(I, l_\infty(M))$ and all $d > 0$, it is satisfied that

$$\|B(x,u)\| < \frac{c+1}{2}\,\|x\| + d \quad \text{for all } \ x \in \mathbb{R}^M \text{ such that } \ \|x\| > K.$$

Therefore, by taking $d > K\,(1-c)/2$, we arrive at the fact that $a = (a_\nu)_{\nu \in M}$ and $b = (b_\nu)_{\nu \in M}$ defined as

$$a_\nu = -\frac{2\,d}{1-c} \qquad \text{and} \qquad b_\nu = \frac{2\,d}{1-c},$$

satisfy the properties imposed in condition $(B2)$.

Thus theorem 3.1 ensures the existence of the extremal solutions, x_* and x^*, in the set

$$Y = \{u \in C(I, L_\infty(M)) : a \le u(t_0) \le b\}.$$

Moreover, if u is any solution of (3.1) in particular $u(t_0) = B(u(t_0), u)$. If $\|u(t_0)\| > K$ then, from the previous arguments, we have that

$$\|u(t_0)\| = \|B(u(t_0), u)\| \le \frac{c+1}{2}\,\|u(t_0)\| + d$$

and then

$$\|u(t_0)\| \le \frac{2\,d}{1-c}.$$

Therefore, $a \le u(t_0) \le b$ and we have that $u \in Y$ and thus $x_* \le u \le x^*$. Then x_* and x^* are the extremal solutions. $\square$

Remark 3.1 *If we define lower and upper solutions, α and β, for problem (3.1) as in [5, p. 47] in the case of one equation, and we assume hypotheses (g0), (g1), (g2), (B0), (B1) and*

($\overline{B}$) *There exists $h \in L^1_+(I)$ such that for all $u \in [\alpha, \beta]$*

$$\|g(t,x,u)\| \le h(t) \quad \text{for a.a. } t \in I \text{ and all } \alpha(t) \le x \le \beta(t),$$

we deduce from [6, theorem 4.1] the existence of extremal solutions in the order interval determined by the lower and the upper solution.

Remark 3.2 *The example given in [1, section 5], which is a modification of the well-known example of Dieudonné, shows that theorem 3.1 is not true, in case $M = \mathbb{N}$, when we replace $l_\infty(\mathbb{N})$ by $c_0(\mathbb{N})$, the set of the sequences that converge to zero.*

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