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A Difference Equation Leading to the Irrationality of $\sqrt{2}$

We provide a fresh proof of a very old and well-known fact: the irrationality of $\sqrt{2}$. To our knowledge, this approach is new; at least, we have not seen it in the outstanding references [**1, 2**], although the flavor of the proof reminds us of [**3**].

Theorem. *The square root of* 2 *is irrational.*

Proof. The characteristic equation associated with the second order linear difference equation

$$x_{n+2} = -2x_{n+1} + x_n, \quad n = 0, 1, 2, \dots, \tag{1}$$

is $r^2 = -2r + 1$, whose solutions are $r_1 = \sqrt{2} - 1$ and $r_2 = -(\sqrt{2} + 1)$. So the general solution of (1) is given by

$$x_n = ar_1^n + br_2^n, \quad a, b \in \mathbb{R}, \quad n = 0, 1, 2, \dots. \tag{2}$$

Assume now that $\sqrt{2}$ is rational, or equivalently, $r_1 = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$. By taking $a = q$ and $b = 0$ in (2), we have that $x_0 = q \in \mathbb{Z}$, $x_1 = p \in \mathbb{Z}$ and then by induction $x_n = -2x_{n-1} + x_{n-2} \in \mathbb{Z}$ for all $n = 2, 3, 4, \dots$.

On the other hand, since $0 < r_1 < 1$ and $q \neq 0$, it follows that $x_n = qr_1^n \neq 0$ for all $n = 0, 1, 2, \dots$ and $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} qr_1^n = 0$, a contradiction. ∎

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