Does Lipschitz with Respect to x Imply Uniqueness for the Differential Equation $y' = f(x, y)$?

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1 INTRODUCTION.

Of course, the answer to the question posed in the title is no, in general, but, surprisingly enough, yes in a significant situation to be detailed later.

This paper aims to bring to the attention of the widest possible mathematical audience an elementary result, based on the theorem of differentiation of inverse functions, which extends the applicability of uniqueness theorems. Loosely speaking, it transforms every uniqueness theorem into an alternative version of it with the roles of the dependent and the independent variables interchanged in the assumptions. A remarkable example of such a theorem is the version of Lipschitz's theorem alluded to in the first paragraph, but there are at least as many new possibilities as old uniqueness theorems.

Next we introduce some notation to discuss properly the question considered. Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$, $f : \mathcal{N} \to \mathbb{R}$ a given mapping, and consider the scalar initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{1.1}$$

We recall that a solution of (1.1) is a function $Y : I \to \mathbb{R}$ which is continuously differentiable on the interval I such that $x_0 \in I$, $Y(x_0) = y_0$, and for all $x \in I$ we have $(x, Y(x)) \in \mathcal{N}$ and $Y'(x) = f(x, Y(x))$. As usual, we say that (1.1) has a unique solution if there exists $\alpha > 0$ with the property that (1.1) has one solution defined on $[x_0 - \alpha, x_0 + \alpha]$ and any other solution $Y : I \to \mathbb{R}$ coincides with it on $I \cap [x_0 - \alpha, x_0 + \alpha]$.

Here and henceforth, we assume that f is continuous on $\mathcal{N}$; thus Peano's theorem guarantees that there exists $\varepsilon > 0$ such that (1.1) has at least one solution defined on the interval $[x_0 - \varepsilon, x_0 + \varepsilon]$.

The scalar version of Peano's theorem was published in 1886, see [11], and was extended to systems in 1890, see [12][1]. At that time Peano was already

[1] Peano wrote that paper using his own logical symbols, which makes it rather difficult to read. Three years later, G. Mie wrote in German a more understandable version of it, see [8].

aware of the fact that some continuous functions f allow (1.1) to have more than one solution. He presented the following example in [12]:

EXAMPLE 1.1. The problem (1.1) for $f(x,y) = 3y^{2/3}$ and $x_0 = 0 = y_0$ has more than one solution. Indeed, one can check by direct computation that $Y_1(x) = 0$ and $Y_2(x) = x^3$ are both solutions defined on the whole real line.

Lavrentieff constructed a more dramatic example in 1925, which consisted in a continuous function on a rectangle such that uniqueness fails for (1.1) at *every* initial condition (x_0, y_0) in the rectangle's interior, see [6]. Later, in 1963, Hartman published in this MONTHLY a simpler example of that type with a function defined on the whole plane, see [5].

The main usefulness of differential equations is that they serve as models that describe mathematically many real phenomena and processes. Especially in those cases the existence of more than one solution is disturbing and misleading, because it produces uncertainty about the behavior of the object that we are studying. Moreover, uniqueness and nonuniqueness also have a number of theoretical implications as, for example, in the study of the qualitative behavior at infinity of global solutions, see [13]. Therefore it is of fundamental importance to have adequate tools to decide whether a concrete problem has a unique solution.

Almost every textbook on ordinary differential equations contains a version of the well-known uniqueness theorem published by Lipschitz in 1877, see [7] and, for instance, the monographs [4, 14]. The following suffices for our purposes in this article:

THEOREM 1.1 (Lipschitz's uniqueness theorem). *Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$.*

If f satisfies a Lipschitz condition with respect to the second variable on $\mathcal{N}$, i.e.,

$$\exists\ L > 0 \quad \textit{such that} \quad (x,y),(x,z) \in \mathcal{N} \Rightarrow |f(x,y) - f(x,z)| \le L|y - z|, \tag{1.2}$$

then (1.1) has a unique solution.

Now, does a Lipschitz condition with respect to the *first* variable imply uniqueness for (1.1)? Example 1.1 shows that this is not true in general, and this is often the end of the question, but we urge the reader to go through the remaining few pages to find out that the answer to our question is positive provided that $f(x_0, y_0) \neq 0$.

2 DOUBLE THE UNIQUENESS THEOREMS THAT YOU KNOW (WITH LITTLE EFFORT!).

We begin this section with the following simple technical remark: if $\mathcal{N}' \subset \mathcal{N}$ is another neighborhood of (x_0, y_0) and the problem

$$y' = f_{|\mathcal{N}'}(x,y), \quad y(x_0) = y_0$$

has a unique solution, then (1.1) has a unique solution (here, $f_{|\mathcal{N}'}$ stands for the restriction of f to $\mathcal{N}'$). This observation guarantees that we can pass, without losing generality, to more convenient smaller neighborhoods when studying uniqueness. In doing so, we avoid some technicalities in the proofs.

The next theorem is the core of the present article and establishes the equivalence between uniqueness for (1.1) and uniqueness for a related reciprocal problem, thus doubling the applicability of uniqueness theorems.

THEOREM 2.1. *Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$.*

If $f(x_0, y_0) \neq 0$ then (1.1) has a unique solution if and only if the problem

$$x' = \frac{1}{f(x,y)}, \quad x(y_0) = x_0 \tag{2.3}$$

has a unique solution.

Proof. Since $f(x_0, y_0) \neq 0$ and f is continuous at (x_0, y_0), there exists a neighborhood of (x_0, y_0) where f has constant sign and is bounded. For simplicity, we assume that f and $1/f$ have constant sign and are bounded on $\mathcal{N}$.

To establish the result we will use the following claim, which is interesting in its own right:

Claim. If Y is a solution of (1.1) then Y^{-1} is a solution of (2.3) and, conversely, if X is a solution of (2.3) then X^{-1} is a solution of (1.1).

Let $Y : I \to \mathbb{R}$ be a solution of (1.1); for all $x \in I$ we have $(x, Y(x)) \in \mathcal{N}$ and $Y'(x) = f(x, Y(x))$, so Y' has constant sign on I and, in particular, it has an inverse $Y^{-1} : Y(I) \to \mathbb{R}$. Let us show that $X = Y^{-1}$ solves (2.3). First, $Y(I)$ is an interval that contains y_0 and $X(y_0) = x_0$; second, we use the theorem of differentiation of inverse functions for all $y \in Y(I)$ to obtain that

$$X'(y) = (Y^{-1})'(y) = \frac{1}{Y'(Y^{-1}(y))} = \frac{1}{f(X(y), y)}.$$

The proof of the converse is analogous, so we omit it, and the claim is proven.

Finally, suppose that Y is the unique solution to (1.1) on the interval $I = [x_0 - \alpha, x_0 + \alpha]$, for some $\alpha > 0$. We are going to prove that Y^{-1} is the unique solution to (2.3) on the interval $J = [y_0 - \beta, y_0 + \beta]$ provided that $\beta > 0$ is so small that $J \subset Y(I)$ and if X is any solution to (2.3) defined on an interval $\tilde{J} \subset J$ then $X(\tilde{J}) \subset I$ (such a choice of β is possible because $1/f$ is bounded on $\mathcal{N}$). Let X be a solution to (2.3) on an interval $\tilde{J} \subset J$. Since X^{-1} is a solution to (1.1) on $X(\tilde{J})$ and $X(\tilde{J}) \subset I$, we conclude that $X^{-1} = Y$ on $X(\tilde{J})$. Hence $X = Y^{-1}$ on $\tilde{J}$. Analogous arguments show that uniqueness for (2.3) implies uniqueness for (1.1). ■

The main theoretical importance in the previous equivalence lies in the fact that the dependent and the independent variables interchange their roles when passing from (1.1) to (2.3), and this has the consequence that assumptions are transferred from one argument to the other.

The plan for generating new uniqueness theorems from old ones in case $f(x_0, y_0) \neq 0$ is very simple now: look for appropriate assumptions on f which imply that (2.3) falls inside the scope of the uniqueness theorem that you choose. We carry out this plan with Lipschitz's theorem in the next section.

3 LIPSCHITZ'S UNIQUENESS THEOREM REVISITED.

This section is devoted to the following version of Lipschitz's uniqueness theorem, which is a straightforward consequence of Theorems 1.1 and 2.1.

THEOREM 3.1. *Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$.*

If $f(x_0, y_0) \neq 0$ and, moreover, f satisfies a Lipschitz condition with respect to the first variable on $\mathcal{N}$, i.e.,

$$\exists\, L > 0 \;\; \textit{such that} \;\; (s, y), (x, y) \in \mathcal{N} \Rightarrow |f(s, y) - f(x, y)| \leq L|s - x|, \tag{3.4}$$

then (1.1) has a unique solution.

Proof. Theorem 2.1 applies because $f(x_0, y_0) \neq 0$, so it suffices to prove that (2.3) has a unique solution. To do so, note that the continuity of f implies that there exists a neighborhood of (x_0, y_0) where $|f| \geq |f(x_0, y_0)|/2 =: r > 0$. For simplicity we assume that this holds on $\mathcal{N}$, so for (x, y), $(s, y) \in \mathcal{N}$ we have

$$\left| \frac{1}{f(x, y)} - \frac{1}{f(s, y)} \right| = \left| \frac{f(s, y) - f(x, y)}{f(x, y) f(s, y)} \right| \leq \frac{L}{r^2} |s - x|,$$

and therefore Theorem 1.1 guarantees that (2.3) has a unique solution (remember that x is the dependent variable in (2.3)). ■

A very useful consequence of Theorem 3.1 concerns differential equations with continuously differentiable right-hand sides.

COROLLARY 3.1. *Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$.*

If $f(x_0, y_0) \neq 0$ and, moreover, $\partial f / \partial x$ is continuous on $\mathcal{N}$, then (1.1) has a unique solution.

Proof. Let $\mathcal{N}'$ be a compact neighborhood of (x_0, y_0) such that $\mathcal{N}' \subset \mathcal{N}$, and let $L > 0$ be an upper bound of $|\partial f / \partial x|$ on $\mathcal{N}'$. Now for $(x, y), (s, y) \in \mathcal{N}'$, $x \neq s$, the mean value theorem guarantees the existence of r, strictly between x and s, such that

$$|f(x, y) - f(s, y)| = \left| \frac{\partial f}{\partial x}(r, y) \right| |x - s| \leq L|x - s|.$$

Hence Theorem 3.1 implies that (1.1) has a unique solution. ■

Example 3.1. The nonlinear problem

$$y' = \cos x + x\sqrt[3]{y}, \quad y(0) = 0$$

has a unique solution by virtue of Corollary 3.1. Notice that the right-hand side of the differential equation does not satisfy the assumptions of Lipschitz's uniqueness theorem on any neighborhood of the initial condition.

An important particular case of the preceding corollary is that of autonomous differential equations. The following classical uniqueness result, which goes back to Peano, see [12], follows immediately from Corollary 3.1.

Corollary 3.2. *Let $\varepsilon > 0$, $g : (y_0 - \varepsilon, y_0 + \varepsilon) \to \mathbb{R}$ continuous, and $x_0 \in \mathbb{R}$.*

If $g(y_0) \neq 0$ then the autonomous problem

$$y' = g(y), \quad y(x_0) = y_0 \tag{3.5}$$

has a unique solution.

Example 3.2. In the nonautonomous case the condition $f(x_0, y_0) \neq 0$ alone is not sufficient for uniqueness for (1.1). As an example observe that the change of variable $y = z + x$ transforms the autonomous problem $dz/dx = 3z^{2/3}, z(0) = 0$, which appears in Example 1.1, into the nonautonomous one

$$y' = 3(y - x)^{2/3} + 1, \quad y(0) = 0,$$

for which the right-hand side does not vanish at the initial data and has $Y_1(x) = x$ and $Y_2(x) = x^3 + x$ as solutions for all $x \in \mathbb{R}$. In this case the right-hand side does not satisfy either (1.2) or (3.4) on any neighborhood of the initial condition.

4 CONCLUDING REMARKS.

1. The claim in the proof of Theorem 2.1 also provides us with an integration method, as (2.3) may be integrable even though (1.1) is not. In fact, it is a well-known trick in the field of differential equations to try and solve $dx/dy = 1/f(x, y)$ instead of $dy/dx = f(x, y)$ whenever the last differential equation is not solvable by elementary methods. It seems however that its underlying theoretical implications in connection with uniqueness were not fully exploited until the last decade.

The claim in the proof of Theorem 2.1 was established in a more general form by the authors in [3], and it was used there to derive a number of consequences, including Theorem 3.1 (Theorem 2.7 in [3]). Later, Cid extended Theorem 3.1 to systems of differential equations in [2].

It was after the publication of [2, 3] that the authors became aware of the work done by Mortici in [9, 10]. As far as we know, Mortici was the first author who deduced Theorem 3.1.

2. Most uniqueness criteria are based on generalizations of the standard Lipschitz's condition (1.2), see [1], or require some monotonicity assumptions such as those in [15, 16]. Theorem 2.1 is the key to establishing alternative versions of all of them with assumptions "transferred from y to x". This yields a lot of new uniqueness theorems that readers may find useful in different situations. A complete account of even the most relevant of them exceeds the objectives of the present article, but we point out as a final example the analog of the so-called Peano's uniqueness criterion (which can be looked up in [1]):

THEOREM 4.1. *Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$.*

If $f(x_0, y_0) \neq 0$ and, moreover, f is nondecreasing with respect to its first variable (i.e., $f(s, y) \leq f(x, y)$ whenever $(s, y), (x, y) \in \mathcal{N}$ and $s \leq x$), then (1.1) has a unique solution.

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