On the uniqueness of fixed points for decreasing operators

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Abstract

In this work we present a necessary and sufficient condition for a decreasing map to have at most one fixed point. Some applications to differential equations are also given.

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1 Introduction

It is well-known that a compact increasing operator $T : [\alpha, \beta] \to [\alpha, \beta]$, where $[\alpha, \beta]$ is a nonempty interval in a Banach space E ordered by a positive cone, has the minimal fixed point u and the maximal fixed point v in $[\alpha, \beta]$, in the sense that every fixed point $x \in [\alpha, \beta]$ satisfies $u \le x \le v$. In [1, theorem 11] additional conditions on T are imposed which guarantee that $u = v$, and therefore the uniqueness of the fixed point is obtained.

For nonmonote mappings Kellog proves in [2] the following theorem which ensures the uniqueness of the fixed point in Schauder's theorem (this result has been generalized by several authors [3, 4, 5], but we present this version for simplicity).

Theorem A *Let X be a real Banach space, $D \subset X$ be an open, bounded, convex subset and $T : \bar{D} \to \bar{D}$ be a compact continuous map which is continuously Fréchet differentiable on D. Suppose that (a) for each $x \in D$, 1 is not an eigenvalue of $T'(x)$, and (b) for each $x \in \partial D$, $x \neq T(x)$. Then T has a unique fixed point.*

In section 2 we study the uniqueness of fixed point for decreasing operators. In particular, we present an elementary criterion which establishes that a decreasing operator T has at most one fixed point if and only if the set of fixed points $Fix(T)$ is directed. By combining this criterion with Schauder's theorem we obtain the following alternative result to Theorem A.

Theorem B *Let E be an ordered Banach space, $D \subset E$ a closed, convex, bounded and nonempty set and $T : D \to D$ a compact operator.*

If T is decreasing and $Fix(T)$ is directed then T has a unique fixed point.

In our work the condition "$Fix(T)$ is directed" is fundamental. It is known that every compact operator $T : D \subset E \to D$, with D and E as in Theorem B, has the minimal and the maximal fixed points if and only if $Fix(T)$ is directed (see [6, Theorem 2.1]). Moreover, if T is decreasing Theorem B asserts the uniqueness of the fixed point.

Whenever E is an usual function space (e.g. $\mathcal{C}^k(\Omega)$, $L^p(\Omega), W^{n,m}(\Omega)$) together with the natural pointwise ordering the solution set $\mathcal{S} \subset E$ of a differential equation between given lower and upper solutions is often directed (see [7]). Thus, if the differential equation may be rewritten as a fixed point equation $x = Tx$, with T decreasing and such that Schauder's theorem applies, then Theorem B implies the uniqueness of the solution for the original differential equation.

As example of the applicability of our results we present in section 3 a uniqueness criterion for a Cauchy problem and another one for a periodic boundary value problem.

2 Main results

Let X be a partially ordered set and $Y \subset X$. We say that Y is *upward directed* if for each pair $y_1, y_2 \in Y$ there exists $y_3 \in Y$ such that $y_1 \leq y_3$ and $y_2 \leq y_3$ and we say that Y is *downward directed* if for each pair $y_1, y_2 \in Y$ there exists $y_4 \in Y$ such that $y_4 \leq y_1$ and $y_4 \leq y_2$. Whenever Y is upward and downward directed we say that Y is *directed*.

An operator $T : D \subset X \to X$ is *decreasing* if $x, y \in D$ with $x \leq y$ implies $Tx \geq Ty$. We denote by $Fix(T)$ the set of fixed points of T, that is

$$Fix(T) = \{x \in D : x = Tx\}.$$

Theorem 2.1 *Let X a partially ordered set and $T : D \subset X \to X$ a decreasing operator. Then T has at most one fixed point if and only if $Fix(T)$ is upward directed.*

Proof. If T has at most one fixed point then obviously $Fix(T)$ is upward directed.

Conversely, assume that $Fix(T)$ is upward directed and that $Fix(T) \neq \emptyset$. Then given $x_1, x_2 \in Fix(T)$ there exists $x_3 \in Fix(T)$ such that $x_1 \leq x_3$ and $x_2 \leq x_3$. Now, since T is decreasing, it follows that

$$x_1 = Tx_1 \geq Tx_3 = x_3 \quad \text{and} \quad x_2 = Tx_2 \geq Tx_3 = x_3.$$

Therefore $x_1 = x_2$, and the proof is complete. □

Remark 2.1 *It is clear that Theorem 2.1 remains true if we change "upward directed" by "downward directed" or by "directed".*

Whenever $Fix(T)$ is not upward directed we cannot ensure in general the uniqueness of the fixed point, as we shown in the following simple example: consider in $\mathbb{R}^2$ the usual componentwise partial ordering and define $T : [-1,1] \times [-1,1] \to [-1,1] \times [-1,1]$ as

$$T(x_1, x_2) = (-x_2, -x_1) \quad \textit{for all } (x_1, x_2) \in [-1,1] \times [-1,1].$$

Then T is decreasing, $Fix(T) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1\}$ is not upward directed, and T has infinitely many fixed points.

In the hypotheses of Theorem 2.1 it is possible that $Fix(T) = \emptyset$. If we combine Theorem 2.1 with, for example, Sadovskii's fixed point theorem (see [8, theorem 11.A]), we obtain the following "proper" uniqueness result, which in particular implies Theorem B at introduction.

Theorem 2.2 *Let E be a Banach space equipped with a partial ordering, $D \subset E$ a closed, convex, bounded and nonempty set and $T : D \to D$ a condensing operator.*

If T is decreasing and $Fix(T)$ is upward directed then T has a unique fixed point.

3 Applications to differential equations

3.1 A uniqueness criterion for a discontinuous Cauchy problem

Let a, $b > 0$, $I = [t_0, t_0 + a]$, $f : I \times [x_0 - b, x_0 + b] \subset \mathbb{R}^2 \to \mathbb{R}$ and consider the Cauchy problem

$$x'(t) = f(t, x(t)) \text{ for a.a. } t \in I, \quad x(t_0) = x_0. \tag{3.1}$$

A *Carathéodory solution* of (3.1) is an absolutely continuous function $x : I \to \mathbb{R}$ such that $x(t) \in [x_0 - b, x_0 + b]$ for all $t \in I$ and which satisfies (3.1).

The following uniqueness result is an extension of [9, Theorem 2.2.1] to the case of Carathéodory solutions.

Theorem 3.1 *Assume there exists $M \geq 0$ such that for a.a. $t \in I$*

$$f(t, x) - f(t, y) \geq M(x - y) \quad \text{if} \quad x_0 - b \leq x \leq y \leq x_0 + b. \tag{3.2}$$

Then, problem (3.1) has at most one Carathéodory solution.

Proof. The problem (3.1) is equivalent to the following one

$$x'(t) - Mx(t) = f(t, x(t)) - Mx(t) \text{ for a.a. } t \in I, \ x(t_0) = x_0, \tag{3.3}$$

and the Carathéodory solutions of (3.3) are the fixed points of the operator $T : D \to \mathcal{C}(I)$ defined as

$$Tx(t) = x_0 e^{M(t-t_0)} + \int_{t_0}^{t} e^{M(t-s)} (f(s, x(s)) - Mx(s)) ds,$$

for all $t \in I$ and $x \in D$, where

$$D := \{x \in \mathcal{C}(I) : x(t) \in [x_0 - b, x_0 + b] \text{ for all } t \in I, \ f(\cdot, x(\cdot)) \text{ is integrable in } I\}.$$

We notice that if a solution of (3.1) exists then $D \neq \emptyset$.

Given $x_1, x_2 \in \mathcal{C}(I)$ we consider the usual partial ordering:

$$x_1 \leq x_2 \quad \text{if and only if} \quad x_1(t) \leq x_2(t) \text{ for all } t \in I.$$

From condition (3.2) we deduce that T is decreasing. Moreover $Fix(T)$ is upward directed because the pointwise maximum of two Carathéodory solutions of (3.1) it is also a Carathéodory solution. Therefore, from Theorem 2.1 it follows that T has at most one fixed point, which is equivalent to say that problem (3.1) has at most one Carathéodory solution in I. $\square$

Remark 3.1 *Observe that f is not assumed to be continuous.*

On the other hand, if for a.a. $t \in I$ the function $f(t,\cdot)$ is decreasing in $[x_0 - b, x_0 + b]$, then f satisfies condition (3.2) for $M = 0$.

3.2 A uniqueness criterion for a periodic boundary value problem

We consider the second order periodic problem

$$u''(t) = f(t, u(t)),\ u(a) = u(b),\ u'(a) = u'(b), \tag{3.4}$$

where $a < b$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function (see the definition in [10]).

To simplify the notations we extend $f(t,x)$ by periodicity, i.e., $f(t,x) = f(t+b-a,x)$ for all $(t,x) \in \mathbb{R}^2$.

A function $\alpha \in \mathcal{C}([a,b])$ such that $\alpha(a) = \alpha(b)$ is a *lower solution* of problem (3.4) if its periodic extension on $\mathbb{R}$ is such that for any $t_0 \in \mathbb{R}$

either $D_-\alpha(t_0) < D^+\alpha(t_0)$,

or there exist an open interval I_0 such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and for a.a. $t \in I_0$,

$$\alpha''(t) \geq f(t, \alpha(t)).$$

A function $\beta \in \mathcal{C}([a,b])$ such that $\beta(a) = \beta(b)$ is an *upper solution* of problem (3.4) if its periodic extension on $\mathbb{R}$ is such that for any $t_0 \in \mathbb{R}$

either $D^-\beta(t_0) > D_+\beta(t_0)$,

or there exist an open interval I_0 such that $t_0 \in I_0$, $\beta \in W^{2,1}(I_0)$ and for a.a. $t \in I_0$,

$$\beta''(t) \leq f(t, \beta(t)).$$

The following result [10, Theorem 1.1] ensures that a solution of (3.4) exists in the sector between a lower and an upper solution.

Theorem 3.2 *Let α and β be lower and upper solutions of (3.4) such that $\alpha \leq \beta$, define*

$$E = \{(t,x) \in [a,b] \times \mathbb{R} \ : \ \alpha(t) \leq x \leq \beta(t))\},$$

and assume that $f : E \to \mathbb{R}$ is a L^1-Carathéodory function.

Then the problem (3.4) has at least one solution $x \in W^{2,1}(a,b)$ such that for all $t \in [a,b]$

$$\alpha(t) \leq x(t) \leq \beta(t).$$

The main idea in the proof of Theorem 3.2 is to show the equivalence between the set of solutions $x \in W^{2,1}(a,b)$ of (3.4) such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in [a,b]$ and the set of fixed points of operator $T : \mathcal{C}([a,b]) \to \mathcal{C}([a,b])$ defined as

$$Tx(t) = \int_a^b G_M(t,s)[f(t, \gamma(s, x(s))) - M\gamma(s, x(s))]ds, \tag{3.5}$$

for all $t \in [a,b]$ and $x \in \mathcal{C}([a,b])$, where $M > 0$, $\gamma : [a,b] \times \mathbb{R} \to \mathbb{R}$ is given by

$$\gamma(t,x) = \begin{cases} \beta(t), & \text{if } x > \beta(t), \\ x, & \text{if } \alpha(t) \le x \le \beta(t), \\ \alpha(t), & \text{if } x < \alpha(t), \end{cases}$$

and $G_M : [a,b] \times [a,b] \to \mathbb{R}$ is the Green's function of problem

$$x''(t) - Mx(t) = f(t),\ x(a) = x(b),\ x'(a) = x'(b). \tag{3.6}$$

Then, since the operator T is completely continuous and bounded, Schauder's fixed point theorem implies that T has a fixed point, which is a solution of (3.4). (In fact in [10] the authors only consider the case $M = 1$, but the same result is true for any $M > 0$).

Whenever $f(t,\cdot)$ is increasing De Coster and Habets proved that there exists a continuum of solutions of problem (3.4) (see [10, Theorem 1.4]). Under a stronger assumption we are going to prove that problem (3.4) has a unique solution between given lower and upper solutions.

Theorem 3.3 *Let α and β be lower and upper solutions of (3.4) such that $\alpha \le \beta$, define*

$$E = \{(t,x) \in [a,b] \times \mathbb{R} \ : \ \alpha(t) \le x \le \beta(t))\},$$

and assume that $f : E \to \mathbb{R}$ is a L^1-Carathéodory function and there exists $M > 0$ such that for a.a. $t \in [a,b]$

$$f(t,x) - f(t,y) \le M(x-y) \quad \text{for all } \alpha(t) \le x \le y \le \beta(t). \tag{3.7}$$

Then the problem (3.4) has a unique solution $x \in W^{2,1}(a,b)$ such that for all $t \in [a,b]$

$$\alpha(t) \le x(t) \le \beta(t).$$

Proof. We define $[\alpha,\beta] := \{x \in \mathcal{C}([a,b]) : \alpha(t) \le x(t) \le \beta(t) \quad \text{for all } t \in [a,b]\}$.

Since the solutions $x \in W^{2,1}(a,b)$ of (3.4) which satisfy $x \in [\alpha,\beta]$ matches up the set of fixed points of T, defined in (3.5), Theorem 3.2 implies that $Fix(T) \ne \emptyset$.

In $\mathcal{C}([a,b])$ we consider the pointwise ordering. Then the following claims hold.

Claim i).- T is decreasing.

Since the Green's function of problem (3.6) satisfies $G_M(t,s) < 0$ for all $(t,s) \in [a,b] \times [a,b]$ (see [11, corollary 2.2]), since γ is increasing and from condition (3.7) it follows that T is decreasing.

Claim ii).- $Fix(T)$ is upward directed.

Let $x_1, x_2 \in Fix(T)$. Then x_1 and x_2 are solutions of (3.4), in particular are lower solutions, which moreover satisfy $x_1, x_2 \in [\alpha,\beta]$. We define

$$\alpha_1(t) := \max\{x_1(t), x_2(t)\} \quad \text{for all } t \in [a,b].$$

By [10, Theorem 1.2] we have that there exists a solution x_3 of (3.4) between α_1 and β, that is, $x_3 \in Fix(T)$ and $\alpha_1 \le x_3 \le \beta$. Therefore, $x_1 \le x_3$ and $x_2 \le x_3$, which means that $Fix(T)$ is upward directed.

Then, Theorem 2.1 ensures that T has a unique fixed point, which is the unique solution of (3.4) between α and β. $\square$

Corollary 3.4 *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a L^1-Carathéodory function such that*

(i) for some $r_1 \leq r_2 \in \mathbb{R}$ and a.a. $t \in [a, b]$ we have that $f(t, r_1) \leq 0 \leq f(t, r_2)$.

(ii) for a.a. $t \in [a, b]$, the function $f(t, \cdot)$ is absolutely continuous and $\frac{d}{dx} f(t, x) \geq M > 0$ for a.a. $x \in [r_1, r_2]$.

Then the problem (3.4) has a unique solution $x \in W^{2,1}(a, b)$ such that for all $t \in [a, b]$

$$r_1 \leq x(t) \leq r_2.$$

Proof. By condition (i) the functions $\alpha(t) = r_1$ and $\beta(t) = r_2$ for all $t \in [a, b]$ are a lower and an upper solutions, respectively, and $\alpha \leq \beta$. Moreover, condition (ii) implies that (3.7) holds. Therefore, the conclusion of the corollary follows from Theorem 3.3. $\square$

Remark 3.2 *Theorem 3.3 asserts the uniqueness of solution of problem (3.4) in the functional interval $[\alpha, \beta]$, but it is possible that another solution $\bar{x}$ of problem (3.4) exists (in this case, of course, $\bar{x} \notin [\alpha, \beta]$).*

For example, consider the problem

$$u''(t) = \sin(u(t)), \; u(0) = u(2\pi), \; u'(0) = u'(2\pi). \tag{3.8}$$

Taking $r_1 = -1$, $r_2 = 1$ and $M = \cos(1) > 0$, Corollary 3.4 is applicable to problem (3.8) and then there exists a unique solution $x \in W^{2,1}(0, 2\pi)$ such that $-1 \leq x(t) \leq 1$ for all $t \in [0, 2\pi]$ (that solution is obviously $x(t) = 0$). Nevertheless, problem (3.8) has infinitely many solutions.

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