On the sign of the Green's function associated to Hill's equation with an indefinite potential*

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Abstract

In this note we give a L^p – criterium for the positiveness of the Green's function of the periodic boundary value problem

$$x'' + a(t)\ x = 0, \quad x(0) = x(T), \quad x'(0) = x'(T),$$

with and indefinite potential $a(t)$. Moreover we prove that such Green's function is negative provided $a(t)$ belongs to the image of a suitable periodic Ricatti type operator.

Keywords. Green's function; anti – maximum principle; Hill's equation.

1 Introduction

Let us say that the linear problem

$$x'' + a(t)\ x = 0, \quad x(0) = x(T), \quad x'(0) = x'(T), \tag{1.1}$$

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is *nonresonant* when its unique solution is the trivial one. It is well known that if (1.1) is nonresonant then, provided that h is a L^1 – function, the Fredholm's alternative theorem implies that the non homogeneous problem

$$x'' + a(t)\, x = h(t), \quad \text{a. e. } t \in [0,T]; \quad x(0) = x(T), \quad x'(0) = x'(T),$$

always has a unique solution which, moreover, can be written as

$$x(t) = \int_0^T G(t,s)h(s)ds,$$

where $G(t,s)$ is the Green's function related to (1.1).

In recent years the condition

(H) Problem (1.1) is nonresonant and the corresponding Green's function $G(t,s)$ is positive (nonnegative) on $[0,T] \times [0,T]$,

has become an standard assumption in the searching for positive solutions of singular second order equations and systems (see for instance [3, 4, 5, 6, 12]). Moreover the positiveness of Green's function implies that an anti – maximum principle holds, which is a fundamental tool in the development of the monotone iterative technique (see [2, 13]).

When $a(t) \equiv k^2$ condition (H) is equivalent to $0 < k^2 < (\leq)\lambda_1 := (\pi/T)^2$, where λ_1 is the first eigenvalue of the homogeneous equation $x'' + k^2 x = 0$ with Dirichlet boundary conditions $x(0) = 0 = x(T)$.

For a non-constant function $a(t)$ the best condition available in the literature implying (H) is a L^p – criterium proved in [10] (and based in an anti – maximum principle given in [13]). For the sake of completeness let us recall such result: define $K(\alpha, T)$ as the best Sobolev constant in the inequality

$$C\|u\|_\alpha^2 \leq \|u'\|_2^2 \quad \text{for all } u \in H_0^1(0,T),$$

given explicitly by (see [9])

$$K(\alpha,T) = \begin{cases} \frac{2\pi}{\alpha T^{1+2/\alpha}} \left(\frac{2}{2+\alpha}\right)^{1-2/\alpha} \left(\frac{\Gamma(1/\alpha)}{\Gamma(1/2+1/\alpha)}\right)^2, & \text{if } 1 \leq \alpha < \infty, \\ \frac{4}{T}, & \text{if } \alpha = \infty. \end{cases} \tag{1.2}$$

Through the paper $a \succ 0$ means that $a \in L^1(0,T)$, $a(t) \geq 0$ for a.a. $t \in [0,T]$ and $\|a\|_1 > 0$, moreover $a_+ = \max\{a, 0\}$ is the positive part of a and for $1 \leq p \leq \infty$ we denote by p^* its conjugate (that is, $\dfrac{1}{p} + \dfrac{1}{p^*} = 1$). Now [10, Corollary 2.3] reads as follows.

THEOREM 1.1. *Assume that $a \in L^p(0,T)$ for some $1 \le p \le \infty$, $a \succ 0$ and moreover*

$$\|a\|_p < (\le) K(2p^*, T).$$

Then condition (H) holds.

Our main goal is to improve Theorem 1.1 by allowing $a(t)$ to change sign. In particular instead of $a \succ 0$ we impose an integral condition, namely $\int_0^T a(t) > 0$, which doesn't prevent $a(t)$ to be negative in a set of positive measure. As far as we are aware this is the first anti – maximum principle for problem (1.1) with an indefinite potential $a(t)$ (compare with the previous results obtained in [1, 10]). Moreover we notice that an improvement of Theorem 1.1 immediately extends the applicability of those results available on the literature which rely on condition (H) as, for instance, the validity of the monotone iterative methods [13], or the existence of constant sign periodic solutions for regular [10, 11], strong singular [4, 6] and weak singular [3, 4, 5, 6, 12] second order boundary value problems.

This paper is organized as follows: in section 2 we present some known results about the Dirichlet, periodic and anti – periodic eigenvalues of equation

$$x'' + (\lambda + a(t))\ x = 0,$$

which are needed on section 3 to prove the positivity of the Green's function of (1.1) with an indefinite potential. In section 4, provided that $\int_0^T a(t)\,dt < 0$, we give a sufficient condition that ensures that the Green's function related to problem (1.1) is negative. Finally, in section 5, we conclude our paper with some remarks referred to the general operator $x'' + c(t)\,x' + a(t)\,x$ with c a L^1 – function with mean value equals to zero.

2 Preliminaries

In this section we collect some known results (see [8]) for the eigenvalue problem

$$x'' + (\lambda + a(t))\ x = 0, \tag{2.1}$$

where $a \in L^1(0,T)$, subject to periodic

$$x(0) = x(T), \quad x'(0) = x'(T), \tag{2.2}$$

anti – periodic

$$x(0) = -x(T), \quad x'(0) = -x'(T), \tag{2.3}$$

or Dirichlet boundary conditions

$$x(0) = 0 = x(T). \tag{2.4}$$

With respect to the periodic and anti – periodic eigenvalues there exist sequences

$$\overline{\lambda}_0(a) < \underline{\lambda}_1(a) \leq \overline{\lambda}_1(a) < \underline{\lambda}_2(a) \leq \overline{\lambda}_2(a) < \ldots < \underline{\lambda}_k(a) \leq \overline{\lambda}_k(a) < \ldots \tag{2.5}$$

such that

(i) λ is an eigenvalue of (2.1) – (2.2) if and only if $\lambda = \underline{\lambda}_k(a)$ or $\overline{\lambda}_k(a)$ for k even.

(ii) λ is an eigenvalue of (2.1) – (2.3) if and only if $\lambda = \underline{\lambda}_k(a)$ or $\overline{\lambda}_k(a)$ for k odd.

On the other hand the Dirichlet problem (2.1) – (2.4) has a sequence of eigenvalues

$$\lambda_1^D(a) < \lambda_2^D(a) < \ldots < \lambda_k^D(a) < \ldots,$$

and the periodic and anti – periodic eigenvalues can be realized for $k = 1, 2, \ldots$ as

$$\underline{\lambda}_k(a) = \min\{\lambda_k^D(a_s) : s \in \mathbb{R}\}, \quad \overline{\lambda}_k(a) = \max\{\lambda_k^D(a_s) : s \in \mathbb{R}\},$$

where $a_s(t) := a(t+s)$ are translations.

In [15, Theorem 4] Zhang and Li established the following lower bound for the first Dirichlet eigenvalue $\lambda_1^D(a)$ in terms of the L^α-norm of a_+.

Theorem 2.1. *Assume that $a \in L^p(0,T)$ for some $1 \leq p \leq \infty$. If*

$$\|a_+\|_p \leq K(2p^*, T),$$

where K is given by (1.2), then

$$\lambda_1^D(a) \geq \left(\frac{\pi}{T}\right)^2 \left(1 - \frac{\|a_+\|_p}{K(2p^*, T)}\right) \geq 0.$$

Note that, since $\underline{\lambda}_1(a) = \lambda_1^D(a_{s_0})$ for some $s_0 \in \mathbb{R}$ and, by considering the T – periodic extension of the function a it is satisfied that $\|(a_{s_0})_+\|_p = \|(a_+)\|_p$, then under the assumptions of Theorem 2.1 we have

$$\underline{\lambda}_1(a) = \lambda_1^D(a_{s_0}) \geq \left(\frac{\pi}{T}\right)^2 \left(1 - \frac{\|a_+\|_p}{K(2p^*, T)}\right) \geq 0. \tag{2.6}$$

3 Positivity of the Green's function

Firstly, we are going to give a sufficient condition for problem (1.1) to be nonresonant which is equivalent to the existence of Green's function.

THEOREM 3.1. *Assume that $a \in L^p(0,T)$ for some $1 \leq p \leq \infty$, $\int_0^T a(t)dt > 0$ and moreover*

$$\|a_+\|_p \leq K(2p^*, T).$$

Then problem (1.1) is nonresonant.

Proof. It is known (see [8]) that

$$\overline{\lambda}_0(a) \leq -1/T \int_0^T a(t)dt < 0.$$

On the other hand, since a satisfies the assumptions of Theorem 2.1, from (2.6) it follows that $\underline{\lambda}_1(a) \geq 0$. Therefore, (2.5) implies that

$$\overline{\lambda}_0(a) < 0 \leq \underline{\lambda}_1(a) < \underline{\lambda}_2(a) \leq \overline{\lambda}_2(a),$$

which means that $\lambda = 0$ is not an eigenvalue of problem (2.1) – (2.2). □

Before to present our main result we need the following auxiliary result (see [10, Theorem 2.1])

LEMMA 3.1. *Assume that (1.1) is nonresonant and that the distance between two consecutive zeroes of a nontrivial solution of*

$$x'' + a(t)\ x = 0$$

is strictly greater that T. Then the Green's function $G(t,s)$ doesn't vanish (and therefore has constant sign).

Now we are going to give a sufficient condition ensuring the positiveness of the Green's function of (1.1) with an indefinite potential $a(t)$. To the best of our knowledge this result is achieved for the first time for a non constant sign potential $a(t)$.

THEOREM 3.2. *Assume that $a \in L^p(0,T)$ for some $1 \leq p \leq \infty$, $\int_0^T a(t)dt > 0$ and moreover*

$$\|a_+\|_p < K(2p^*, T).$$

Then $G(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$.

Proof. Claim.- The distance between two consecutive zeroes of a nontrivial solution of $x'' + a(t)\ x = 0$ is strictly greater that T.

To the contrary assume that x is a nontrivial solution of the Dirichlet problem

$$x''(t) + \tilde{a}(t)\, x(t) = 0,\ t \in [t_1, t_2], \quad x(t_1) = 0 = x(t_2), \tag{3.1}$$

where $0 < t_2 - t_1 \leq T$ and $\tilde{a}$ is the restriction of function a to the interval $[t_1, t_2]$. It is clear, from expression (1.2), that for any α fixed, the expression $K(\alpha, T)$ is strictly decreasing in $T > 0$. As consequence, since $0 < t_2 - t_1 \leq T$, we deduce the following properties:

$$\|\tilde{a}_+\|_p \leq \|a_+\|_p < K(2p^*, T) \leq K(2p^*, t_2 - t_1).$$

From Theorem 2.1 it follows that

$$\lambda_1^D(\tilde{a}) > 0,$$

which contradicts that (3.1) has a nontrivial solution.

Now, Lemma 3.1 and the *claim* imply that $G(t,s)$ doesn't vanish. To determinate its sign consider the periodic problem

$$x''(t) + a(t)\ x(t) = 1, \quad x(0) = x(T), \quad x'(0) = x'(T).$$

It is clear that its unique solution is given by the expression

$$x(t) = \int_0^T G(t,s) ds.$$

Obviously x doesn't vanish and has the same sign as G. Then, dividing the equation by x and integrating over $[0, T]$ we obtain

$$0 < \int_0^T \left(\frac{x'(t)}{x(t)} \right)^2 dt + \int_0^T a(t) dt = \int_0^T \frac{dt}{x(t)}.$$

Hence $x(t) > 0$ on $[0, T]$ which implies $G(t,s) > 0$ on $[0, T] \times [0, T]$. □

EXAMPLE 3.1. As a direct consequence of the previous result, we deduce that for any $c > 0$ and $h \in L^1(0, 2\pi)$, if $\|(c + b\ \cos t)_+\|_p < K(2p^*, 2\pi)$, then the following equation

$$x''(t) + (c + b\ \cos t)\, x(t) = h(t),$$

has a unique 2π – periodic solution. Moreover, if h has constant sign then $x(t)\, h(t) \geq 0$ for all $t \in [0, 2\pi]$.

4 Negativeness of the Green function

When $a \prec 0$ it is known that $G(t,s) < 0$. In this section we present a sufficient condition that ensures us the negativeness of the Green's function even in the case of $a(t)$ changes sign. As far as the authors are aware this is the first result in this direction for an indefinite potential $a(t)$.

THEOREM 4.1. *Assume that $a \in L^1(0,T)$ is of the form*

$$a(t) = b'(t) - b^2(t), \quad b(0) = b(T),$$

where b is an absolutely continuous function such that $\int_0^T b(s)ds \neq 0$.

Then $G(t,s) < 0$ for all $(t,s) \in [0,T] \times [0,T]$.

Proof. The key idea is to decompose the second order operator

$$L\,x = x'' + a(t)\,x,$$

as two first order operators $L = L_1 \circ L_2$, where

$$L_1\,x = x' - b(t)\,x \quad \text{and} \quad L_2\,x = x' + b(t)\,x.$$

The following claim is easily proved by direct integration.

Claim.- The problem $x' + b(t)x = h$, $x(0) = x(T)$, has a unique solution for all $h \in L^1(0,T)$ if and only if $\int_0^T b(s)ds \neq 0$. Moreover if $h \succ (\prec) 0$ then $x(t)\int_0^T b(s)ds > (<) 0$ for all $t \in [0,T]$.

Now, suppose that $\int_0^T b(s)ds > 0$ (the other case being analogous). If $Lx \succ 0$, $x(0) = x(T)$, $x'(0) = x'(T)$ then $L_1(L_2 x) \succ 0$ with $L_2 x(0) = L_2 x(T)$ and from the *claim* it follows that $L_2 x < 0$. Now the *claim* implies again that $x < 0$. This fact is equivalent to the negativeness of the Green's function, concluding the proof. □

REMARK 4.1. Note that the previous result extends for the non constant potential $a(t)$ the classical one in which $a(t) \equiv a < 0$ is a strictly negative constant. Moreover we remark that assumptions of Theorem 4.1 imply $\int_0^T a(t)dt < 0$.

Example 4.1. As a direct consequence of the previous result, we deduce that for any $n \in \mathbb{N}$, $c \in \mathbb{R}\backslash\{0\}$ and $h \in L^1(0, 2\pi)$, the following equation

$$x''(t) + (\pm n \sin n t - (\mp \cos n t + c)^2)\, x(t) = h(t),$$

has a unique 2π – periodic solution. Moreover, if h has constant sign then x has the opposite one.

5 The general second order operator

In this section we extend Theorems 3.2 and 4.1 to the general second order equation

$$u'' + c(t)\, u' + a(t)\, u = h(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{5.1}$$

where $c \in L^1(0,T)$ is a damping time – dependent coefficient with mean value zero, i.e. $\int_0^T c(s) ds = 0$.

Let us define for all $t \in [0,T]$ the functions

$$\rho(t) = e^{\int_0^t c(s) ds} \quad \text{and} \quad w(t) = \int_0^t \frac{ds}{\rho(s)},$$

and denote $R = w(T)$.

Thus we obtain the following result.

Theorem 5.1. *Assume that $c \in L^1(0,T)$ with $\int_0^T c(s)\, ds = 0$, $a \in L^p(0,T)$ for some $1 \le p \le \infty$, $\int_0^T \rho(t)\, a(t)\, dt > 0$ and moreover*

$$\|\rho^{\frac{2p-1}{p}} a_+\|_{L^p[0,T]} < K(2p^*, R).$$

Then $G(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$, where G is the Green's function of problem (5.1).

Proof. Making the change of variables $x(r) = u(w^{-1}(r))$ for all $r \in [0,R]$ we have that

$$x'(r) = u'(w^{-1}(r))\rho(w^{-1}(r)),$$

and

$$x''(r) = u''(w^{-1}(r))\rho^2(w^{-1}(r)) + u'(w^{-1}(r))\rho^2(w^{-1}(r))c(w^{-1}(r)).$$

Thus, it is easy to check that if u is a solution of problem (5.1) then $x(r) = u(w^{-1}(r))$ is a solution of problem

$$x''(r) + \rho^2(w^{-1}(r))a(w^{-1}(r))x(r) = \rho^2(w^{-1}(r))h(w^{-1}(r)),\ r \in [0, R], \quad (5.2)$$

$$x(0) = x(R), \quad x'(0) = x'(R), \quad (5.3)$$

and reciprocally, if x is a solution of (5.2) – (5.3) then $u(t) = x(w(t))$ is a solution of (5.1).

On the other hand, the linear left-hand side of equation (5.2) is a Hill's equation of the form $x''(r) + \tilde{a}(r)\,x(r)$ with

$$\tilde{a}(r) = \rho^2(w^{-1}(r))\,a(w^{-1}(r)).$$

From our assumptions it follows that

$$\int_0^R \tilde{a}(r)dr = \int_0^T \rho(s)\,a(s)ds > 0,$$

and

$$\|\tilde{a}_+\|_{L^p[0,R]} = \|\rho^{\frac{2p-1}{p}}\,a_+\|_{L^p[0,T]} < K(2p^*, R).$$

Therefore we can apply Theorem 3.2 to ensure that $\tilde{G}(r, s) > 0$ for all $(r, s) \in [0, R] \times [0, R]$, where $\tilde{G}$ is the Green's function related to problem (5.2)-(5.3), and we also know that its unique solution is given by

$$x(r) = \int_0^R \tilde{G}(r, s)\,\rho^2(w^{-1}(s))\,h(w^{-1}(s))\,ds, \quad \text{for all } r \in [0, R].$$

Thus the unique solution of (5.1) under our assumptions is given by

$$u(t) = x(w(t)) = \int_0^T \tilde{G}(w(t), w(s))\,\rho(s)\,h(s)\,ds, \quad \text{for all } t \in [0, T].$$

This last equation implies that the Green's function related to problem (5.1) is equals to

$$G(t, s) = \tilde{G}(w(t), w(s))\,\rho(s) \quad \text{for all } (t, s) \in [0, T] \times [0, T],$$

and hence $G(t, s) > 0$ for all $(t, s) \in [0, T] \times [0, T]$. □

Remark 5.1. In Theorem 5.1 we have used a change of variables different from the standard one, namely $u(t) = e^{-\frac{1}{2}\int_0^t c(s)ds}x(t)$, since it allows us to impose less restrictive conditions over the function $c(t)$.

Although the assumption $\int_0^T c(s)ds = 0$ does not seem to have a physical meaning, from the mathematical point of view our result complements [11, Corollary 2.5], where the author established a L^p– maximum principle for problem (5.1) with a constant positive coefficient $c(t) \equiv c > 0$. Moreover it gives additional information to the one proved in [1] for the general operator of second order coupled with different kinds of boundary conditions. There two cases were considered: $a < 0$ or a positive and bounded with c bounded.

A related maximum principle for the general second order operator (5.1), with a damped coefficient $c(t)$ without necessarily mean value zero, was proved in [14] and used in the recent paper [7] to prove the existence of a periodic solution for a differential equation with a weak singularity.

In an similar way to the previous result, we deduce the following one as a direct consequence of Theorem 4.1.

THEOREM 5.2. *Assume that* $c \in L^1(0,T)$ *with* $\int_0^T c(s)ds = 0$, $a \in L^1(0,T)$ *satisfies*

$$\rho^2(t)\, a(t) = \rho(t)\, b'(t) - b^2(t), \quad b(0) = b(T),$$

with b *an absolutely continuous function such that* $\int_0^T \frac{b(s)}{\rho(s)} ds \neq 0$.

Then $G(t,s) < 0$ *for all* $(t,s) \in [0,T] \times [0,T]$, *where* G *is the Green's function of problem (5.1).*

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