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HETEROCLINIC SOLUTIONS FOR NON-AUTONOMOUS BOUNDARY VALUE PROBLEMS WITH SINGULAR Φ-LAPLACIAN OPERATORS

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Abstract. We prove the solvability of the following boundary value problem on the real line

$$\begin{cases} \Phi(u'(t))' = f(t,u(t),u'(t)) \quad \text{on } \mathbb{R}, \\ u(-\infty) = -1, \quad u(+\infty) = 1, \end{cases}$$

with a singular Φ-Laplacian operator.

We assume f to be a continuous function that satisfies suitable symmetry conditions. Moreover some growth conditions in a neighborhood of zero are imposed.

1. Introduction.

The study of the existence of travelling wave solutions for reaction-diffusion equations has motivated in the recent years many papers concerning the existence of heteroclinic solutions for second order equations (see for instance [1, 10, 11, 13]).

In the recent paper [4] Bianconi and Papalini study the non-autonomous problem

$$\begin{cases} \Phi(u'(t))' = f(t,u(t),u'(t)), \quad \text{a.e. on } \mathbb{R}, \\ u(-\infty) = 0, \quad u(+\infty) = 1, \end{cases}$$

where the usual linear second order operator u'' is replaced by the nonlinear one $\Phi(u'(t))'$. Here $\Phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\Phi(0) = 0$. The paradigm for this operator is the classical one-dimensional p-Laplacian

$$\Phi_p(s) = |s|^{p-2}s, \quad p > 1.$$

The p-Laplacian operator arises in non-Newtonian fluid theory (as well as in the diffusion of flows in porus media or in nonlinear elasticity) and has became a very popular subject in the last decades (see [8, 12, 15, 14] and references therein). Some existence results for the p-Laplacian in the presence of lower and upper solutions were extended for arbitrary increasing homeomorphisms Φ with different kinds of boundary conditions in [5, 6].

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Recently some papers have appeared where the authors consider Φ-Laplacian type equations with homeomorphisms $\Phi : (-a, a) \to (-b, b)$ for $0 < a, b \le +\infty$ (see [2, 3, 7, 9]). When $b < +\infty$ the Φ-Laplacian is said to be *bounded* or *non-surjective* and the classical model is the mean curvature operator $\Phi(s) = \frac{s}{\sqrt{1+s^2}}$ for $s \in \mathbb{R}$. On the other hand if $a < +\infty$ then the Φ-Laplacian is said to be *singular*, in the terminology of Bereanu and Mawhin [3], and in this case the model is the relativistic operator $\Phi(s) = \frac{s}{\sqrt{1-s^2}}$ for $s \in (-1, 1)$.

In this paper we contribute to the literature studying the following boundary value problem on the real line

$$\begin{cases} \Phi(u'(t))' = f(t, u(t), u'(t)), & \text{on } \mathbb{R}, \\ u(-\infty) = -1, \quad u(+\infty) = 1, \end{cases}$$

where Φ is singular.

In [3] Bereanu and Mawhin have proven the striking result that for a singular Φ-Laplacian the Dirichlet problem

$$\begin{cases} \Phi(u'(t))' = f(t, u(t), u'(t)), & \text{for all } t \in [0, T], \\ u(0) = 0 = u(T), \end{cases}$$

is always solvable for every continuous function f and every $T > 0$ without additional assumptions (see also [7]). This "universal" solvability is related with the fact that all solutions of this problem have their derivatives a priori bounded. In this paper we exploit this fact in order to perform an approximation procedure to deal with our infinite interval problem.

2. **Preliminaries.**

We shall deal with the problem

$$\Phi(u'(t))' = f(t, u(t), u'(t)) \quad \text{on } \mathbb{R}, \tag{1}$$

$$u(-\infty) = -1, \quad u(+\infty) = 1, \tag{2}$$

under the following assumptions:

(h0) $\Phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism, with $\Phi(0) = 0$ and $0 < a < +\infty$ (i.e., Φ is singular).

(f0) $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous and satisfies the symmetry condition

$$f(t, x, y) = -f(-t, -x, y) \quad \text{for all } (t, x, y) \in \mathbb{R}^3.$$

(f1) $f(t, 1, y) = 0 = f(t, -1, y)$ for all t, $y \in \mathbb{R}$.

(f2) $f(t, x, y) < 0$ for all $t > 0$, $-1 < x < 1$ and $y \in \mathbb{R}$. Moreover for every compact set of the form $K = [-r, r] \times [-\varepsilon, \varepsilon]$, where $0 < r < 1$ and $0 < \varepsilon < 1$, there exist $t_K \ge 0$ and a continuous function $h_K : [t_K, \infty) \to \mathbb{R}$ such that

$$f(t, x, y) \le h_K(t) \quad \text{for all } t \ge t_K \text{ and } (x, y) \in K,$$

and

$$\int_{t_K}^{+\infty} h_k(s)ds = -\infty.$$

A *solution* of (1)-(2) is a function $u \in C^1(\mathbb{R})$ such that $u' \in (-a, a)$, $\phi \circ u' \in C^1(\mathbb{R})$ and u satisfies the differential equation (1) and the boundary conditions (2).

We shall approximate problem (1)-(2) by problems defined on compact intervals. The following result shall be very useful for us.

Theorem 2.1. [3, Corollary 1] *Suppose that $\Phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism with $0 < a < +\infty$ and $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Then the Dirichlet problem*

$$\Phi(u'(t))' = f(t, u(t), u'(t)), \quad u(0) = 0 = u(T),$$

has at least one solution. (Notice that in particular $\|u'\|_\infty < a$).

3. **Main results.** Next we prove the solvability of our problem.

Theorem 3.1. *If conditions (h0), (f0), (f1) and (f2) hold then problem (1)-(2) has an odd increasing solution $u : \mathbb{R} \to \mathbb{R}$.*

Proof. By the symmetry condition of $(f0)$ it suffices to prove the existence of a solution $u : [0, +\infty) \to \mathbb{R}$ of (1) satisfying $u(0) = 0$ and $\lim_{t\to+\infty} u(t) = 1$, since its odd extension solves (1)-(2).

Claim 1.- For each $n \in \mathbb{N}$ the Dirichlet boundary value problem

$$\Phi(u'(t))' = f(t, u, u'),\ u(0) = 0,\ u(n) = 0, \tag{3}$$

has a solution $u_n : [0, n] \to \mathbb{R}$ satisfying $0 \le u_n(t) \le 1$ and $\|u_n'\|_\infty < a$.

Consider the continuous function

$$\tilde{f}(t, x, y) = \begin{cases} f(t, x, y), & \text{if } -1 \le x \le 1 \\ 0, & \text{in other case.} \end{cases}$$

For each $n \in \mathbb{N}$ the modified problem

$$\Phi(u'(t))' = \tilde{f}(t, u(t), u'(t)),\ u(0) = 0 = u(n),$$

has by Theorem 2.1 a solution $u_n : [0, n] \to \mathbb{R}$ with $\|u_n'\|_\infty < a$. Moreover it is easy to show that $-1 \le u_n(t) \le 1$ and therefore u_n is also a solution of (3). On the other hand $(f2)$ implies that u_n is concave and then $0 \le u_n(t) \le 1$.

Claim 2.- There exists a bounded nondecreasing solution $u : [0, +\infty) \to \mathbb{R}$ of (1) such that $u(0) = 0$ and $0 \le u(t) \le 1$.

Since u_n and u_n' are uniformly bounded then it is easy to prove that a subsequence of u_n converges uniformly on compact sets to a solution $u : [0, +\infty) \to \mathbb{R}$ of (1). Clearly $u(0) = 0$ and $0 \le u(t) \le 1$.

On the other hand, from the uniform continuity of function Φ^{-1} on compact sets it follows that the sequence $\{u_n'\}$ is an equicontinuous family, and as consequence it is verified that $\Phi(u'(t))' = f(t, u(t), u'(t)) \le 0$. So we deduce that u' is nonincreasing. If $u'(t_0) < 0$ at some point $t_0 \ge 0$ then $u'(t) \le u'(t_0) < 0$ for all $t \ge t_0$ and consequently $\lim_{t\to+\infty} u(t) = -\infty$, a contradiction. Thus $u'(t) \ge 0$ for all $t \ge 0$ and then u is nondecreasing.

Claim 3.- $\lim_{t\to+\infty} u'(t) = 0$.

Since u' is decreasing there exists $\lim_{t\to+\infty} u'(t) \in \mathbb{R} \cup \{-\infty\}$. But as u is bounded we deduce that $\lim_{t\to+\infty} u'(t) = 0$.

Claim 4.- $\lim_{t\to+\infty} u(t) = 1$.

Since u is concave and bounded there exists $\lim_{t\to+\infty} u(t) = l \in (0,1]$. Suppose that $l < 1$. From (f2) and the facts that $0 \le u(t) \le l < 1$ and $\lim_{t\to+\infty} u'(t) = 0$ it follows that there exist a suitable compact set $K \subset (-1,1)\times\mathbb{R}$, $t_K > 0$ for which $0 < u'(t_K) < 1$ and a continuous function h_K such that

$$\Phi(u'(t))' = f(t,u(t),u'(t)) \le h_K(t) \quad \text{for all } t \ge t_K,$$

and $\int_{t_K}^{\infty} h_K(t) = -\infty$. But in this case $\Phi(u'(t)) \to -\infty$ and then $u'(t) \to -a < 0$, which is a contradiction. Thus $l = 1$ and the proof is over. □

Remark 1. With some technical minor modifications the result of Theorem (3.1) also holds for L^1-Carathéodory nonlinearities instead of continuous ones.

Example 1. Let $n \in \mathbb{N}$ be given. Consider the problem

$$\begin{cases} \left(\frac{u'(t)}{\sqrt{1-u'(t)^2}}\right)' = t^3(u(t)^2-1)(u'(t)^{2\,n}+1), & \text{on } \mathbb{R}, \\ u(-\infty) = -1, \quad u(+\infty) = 1, & \end{cases}$$

where $\Phi(s) = \frac{s}{\sqrt{1-s^2}}$ for all $s \in (-1,1)$ models mechanical oscillations subject to relativistic effects and $f(t,x,y) = t^3(x^2-1)(y^{2\,n}+1)$. Clearly conditions of Theorem 3.1 are fulfilled and so its solvability is guaranteed.

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